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Reduced order output feedback control design for PDE systems using proper orthogonal decomposition and nonlinear semidefinite programming

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Abstract

The design of an optimal (output feedback) reduced order control (ROC) law for a dynamic control system is an important example of a difficult and in general non-convex (nonlinear) optimal control problem. In this paper we present a novel numerical strategy to the solution of the ROC design problem if the control system is described by partial differential equations (PDE). The discretization of the ROC problem with PDE constraints leads to a large scale (non-convex) nonlinear semidefinite program (NSDP). For reducing the size of the high dimensional control system, first, we apply a proper orthogonal decomposition (POD) method to the discretized PDE. The POD approach leads to a low dimensional model of the control system. Thereafter, we solve the corresponding small-sized NSDP by a fully iterative interior point constraint trust region (IPCTR) algorithm. IPCTR is designed to take advantage of the special structure of the NSDP. Finally, the solution is a ROC for the low dimensional approximation of the control system. In our numerical examples we demonstrate that the reduced order controller computed

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from the small scaled problem can be used to control the large scale approximation of the PDE system.

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1. Introduction

Optimal control problems for nonlinear partial differential equations are often hard to tackle numerically so that the need for developing novel techniques emerges. One such technique is given by reduced order methods. Recently the application of reduced-order models to optimal control problems for partial differential equations has received an increasing amount of attention. The reduced-order approach is based on projecting the dynamical system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to e.g. finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate. The reduced basis method as developed, e.g., in [23] is one such reduced-order method with the basis elements corresponding to the dynamics of expected control regimes.

Proper orthogonal decomposition (POD) provides a method for deriving low order models of dynamical systems. It was successfully used in a variety of fields including signal analysis and pattern recognition (see [16]), fluid dynamics and coherent structures (see [6,43]) and more recently in control theory (see [1,3,37,42,45]) and inverse problems (see [5]). Moreover, in [4] POD was successfully utilized to compute reduced-order controllers. The relationship between POD and balancing was considered in [30]. Error analysis for nonlinear dynamical systems in finite dimensions were carried out in [21,40].

In our application we apply POD to derive a Galerkin approximation in the spatial variable, with basis functions corresponding to the solution of the physical system at pre-specified time instances. These are called the snapshots. Due to possible linear dependence or almost linear dependence, the snapshots themselves are not appropriate as a basis. Rather a singular value decomposition (SVD) is carried out and the leading generalized eigenfunctions are chosen as a basis, referred to as the POD basis; see Section 2.2. In Section 2.3 the close relationship between POD and SVD is shown. It appears that [28] was the first work that deals with approximation properties of Galerkin POD based schemes for linear as well as certain nonlinear evolution systems. Asymptotic results in the sense that the constants appearing in the estimates did not depend on the snapshot set were discussed in [29].

Output feedback controller synthesis for linear time invariant control systems that meet desired performance and/or robustness specifications is an attractive model-based control design tool and has been an active research area of the control community for several decades (see [11,49]). Indeed, it is not always possible to have full access to the state of the control system and a controller based on the measurements has to be used. Output feedback synthesis without additional complexity constraints yields in general a controller order equal to n the dimension of the dynamical system. The computation of the controller action becomes more expensive with increasing controller dimension. This is one reason why full order synthesis control has not been widely used in industry. In view of real-time implementation the need for fixed output feedback *reduced order control* design is obvious. Furthermore, they are relevant when a simple controller must be used due to cost and reliability. During the past decade, control problems with combined $\mathscr{H}_2$ and $\mathscr{H}_\infty$ design criteria have gained a great deal of attention. Concerning continuous-time systems, [14] provides the solution of standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems in terms of algebraic Riccati equations, where both state feedback and full order compensator-based output feedback are considered. The design of feedback controllers that satisfy both $\mathscr{H}_\infty$ and $\mathscr{H}_2$ specifications is interesting because it offers robust stability and nominal performance. In 1989, Bernstein and Haddad [7] introduced a mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problem. Other related works on the design of $\mathscr{H}_2/\mathscr{H}_\infty$ controllers by state or full order output feedback can be found, for example, in [13,20]. Recently, linear matrix inequalities (LMIs) have attained much attention in control engineering [8], since many control problems can be formulated in terms of LMIs and thus solved via convex programming approaches. For example, this includes $\mathscr{H}_\infty$ [17], $\mathscr{H}_2$ [49], and mixed $\mathscr{H}_2/\mathscr{H}_\infty$ [26]. However, the resulting controllers are state feedback or of order n equal to the plant. The difficulties arise, if we want to design a ROC of given dimension $n_c \geqslant 0$, where $n_c \ll n$. In this case, the optimal control problems consist of finding a ROC law which minimizes a certain nominal performance measure subject to stability and/or robustness constraints. It is known that they can be rewritten to non-convex equality constraint matrix optimization problems (see [22,24,31,32,44]). Notice that in [31] the author extend these matrix optimization problems to NSDPs by including explicitly the stability condition (modelled by two matrix inequalities) into the problem formulation.

Finding a numerical solution to the non-convex NSDP is a difficult task, particularly, if the dimension of the NSDP is large. Usually this will be the case if the control system dynamics are given by a partial differential equation . Then, the dimension of the discretized counterpart can be very large and the computation of an output feedback controller maybe impossible. In particular, the ROC case requires the solution of a large scale NSDP with several million variables, which is usually not solvable. This is one of our main motivations for considering the ROC problem for PDE constrained control problems in combination with the POD method for deriving a low dimensional control system and the IPCTR algorithm for solving the corresponding low dimensional NSDP. The ROC law can be constructed from the solution of low dimensional NSDP. In our numerical examples, we will demonstrate that this ROC can be used for controlling the large dimensional PDE system. To our knowledge, there are only two interior point algorithms available for solving non-convex NSDPs: the QQP Algorithm proposed by Jarre [25] and IPCTR proposed by Leibfritz and Mostafa [34]. The major drawback of the QQP-method is the use of the QR-decomposition of the Jacobian of the nonlinear equality constraints for computing a search direction and the evaluation of the Hessian matrix in every QQP-iteration which can be very time consuming. In [22], the authors have used the QQP method for computing an optimal $\mathscr{H}_\infty$ controller of order $n_c = 5$ for an active suspension system of order $n = 27$. For this example, QQP requires more than four hours for solving the corresponding NSDP. This bad performance of QQP was one of the motivation of the authors in [34] for the development of IPCTR. In particular, IPCTR uses ideas of interior point methods for nonlinear programs and for linear SDPs combined with trust region techniques (see [2,10,12,36,46]). IPCTR exploits the inherent structure of NSDPs arising in the design of static output feedback control problems without evaluating the Hessian matrix explicitly. The computational performance of IPCTR seems very promising, even for moderate sized problems up to several hundred state variables of the control system. To test the IPCTR algorithm, the author in [33] has collected a set of benchmark examples. Test runs of IPCTR on a subset of *COMPl$_e$ib* indicates that IPCTR seems very robust and potentially efficient for the solution of NSDPs arising in the design of ROC or SOF controllers.

One of our main goals in this paper is to demonstrate that a clever combination of efficient solvers is a very useful tool for the design of a feedback control strategy for large scale optimal control problems. In particular, we will use the POD method for reducing the high dimensional (discrete) PDE system to a very low dimensional model. Then, from the small-scaled approximation of our control system, we will form a nonlinear semidefinite program for the computation of the fixed order output feedback controller. As a solver for the NSDP we will take the IPCTR procedure. Finally, having found the ROC law for the small system by IPCTR, we will form the feedback controller for the large system from the information of the low dimensional model. In our numerical studies we will illustrate the applicability of the novel tool for some instances of nonlinear and unstable PDE systems. For our examples, we observe that the feedback control law, computed from the low dimensional approximation of the system, will be able to stabilize the original unstable large dimensional control system quite well.

The organization of the paper is as follows. In Section 2 we state the POD method for abstract dynamical systems. Moreover, the relation of POD to SVD can be found in this paragraph. The discussion of fixed order output feedback control design problems for finite dimensional systems is contained in Section 3. We present three important ROC design instances: $\mathscr{H}_2$, $\mathscr{H}_\infty$ and mixed $\mathscr{H}_2/\mathscr{H}_\infty$ ROC design, respectively, and we derive the corresponding ROC-NSDPs. In Section 4 we state the necessary modification of [34, IPCTR] such that IPCTR is also applicable to the ROC-NSDPs. Finally, we present our numerical results in Section 5.

`Notation:` Throughout this paper, $\mathscr{S}^n$ denotes the linear space of real symmetric $n \times n$ matrices. In the space of real $m \times n$ matrices we define the inner product by $\langle M, Z\rangle = \mathrm{Tr}(M^{\mathrm{T}}Z)$ for $M, Z \in \mathbb{R}^{m\times n}$, where $\mathrm{Tr}(\cdot)$ is the trace operator, and $\|\cdot\|$ denotes the Frobenius norm given by $\|M\| = \langle M, M\rangle^{1/2}$, while other norms and inner products will be specified, e.g., the L^2-norm $\|\cdot\|_{L^2}$. For a matrix $M \in \mathscr{S}^n$ we use the notation $M \succ 0$, $M \succeq 0$, $M \prec 0$, $M \preceq 0$ if it is positive definite, positive semidefinite, negative definite, negative semidefinite, respectively. For a twice differentiable mapping $G : \mathscr{U} \to \mathscr{W}$ we denote by G_U, G_{UU} the first and second partial derivatives of G with respect to U. Moreover, $G_U(\cdot)H$ is used when a linear operator $G_U(\cdot)$ is applied to an element $H \in \mathscr{U}$. Furthermore, $G_U^*(\cdot)$ denotes the adjoint of $G_U(\cdot)$ and $\mathscr{L}(\mathscr{U}, \mathscr{V})$ refers to the space of linear, bounded operators. For related positive quantities α and β, we write $\alpha = \mathscr{O}(\beta)$ if there is a constant $\kappa > 0$ such that $\alpha \geqslant \kappa\beta$ for all β sufficiently small. We also write $\alpha = \Omega(\beta)$ if $\beta = \mathscr{O}(\alpha)$.

2. The POD method

POD is a method to derive reduced-order models for dynamical systems. In this section we introduce the POD method for an abstract dynamical system and propose the numerical realization of POD. The close connection between POD and singular value decomposition (SVD) is shown.

2.1. Abstract dynamical system

Let us consider the following semi-linear initial value problem:

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \mathscr{A}y(t) = f(t, y(t)) & \text{for } t \in (0, T), \\ y(0) = y_\circ, \end{cases} \tag{2.1}$$

where $-\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t > 0$, on X, $y_\circ \in X$ and $f : [0, T] \times X \to X$ is continuous in t and uniformly Lipschitz-continuous on X for every t. Problem (2.1) has a unique mild solution $y \in C([0, T]; X)$ given by the integral representation

$$y(t) = S(t)y_\circ + \int_0^t S(t - s)f(s, y(s))\,\mathrm{d}s \quad \text{for } t \in (0, T),$$

see for instance [39, p. 184]. Then for given time instances $0 \leqslant t_1 < \cdots < t_n \leqslant T$ the members of the ensemble can be given by the mild solution to (2.1):

$$y_j = y(t_j) = S(t_j)y_\circ + \int_0^{t_j} S(t_j - s)f(s, y(s))\,\mathrm{d}s \in X \quad \text{for } j = 1, \ldots, m.$$

2.2. Computation of the POD basis

Let V and H be real separable Hilbert spaces and suppose that V is dense in H with compact embedding. Throughout we assume that X denote either the space V or H and that y denotes the unique solution to (2.1). For given $n \in \mathbb{N}$ let

$$0 = t_1 < t_2 < \cdots < t_n \leqslant T \tag{2.2}$$

denote a grid in the interval $[0, T]$ and set $\delta t_j = t_j - t_{j-1}$, $j = 2, \ldots, n$. We suppose that the *snapshots* $y(t_j)$ of (2.1) at the given time instances t_j, $j = 1, \ldots, n$, are known. We set

$$\mathscr{V} = \operatorname{span}\{y_1, \ldots, y_n\}$$

and refer to $\mathscr{V}$ as the ensemble consisting of the snapshots $\{y_j\}_{j=1}^n$, at least one of which is assumed to be nonzero. Notice that $\mathscr{V} \subset V$ by construction. Throughout the remainder of this section let X denote either the space V or H.

Let $\{\psi_i\}_{i=1}^d$ denote an orthonormal basis for $\mathscr{V}$ with $d = \dim \mathscr{V}$. Then each member of the ensemble can be expressed as

$$y_j = \sum_{i=1}^{d} \langle y_j, \psi_i \rangle_X \psi_i \quad \text{for } j = 1, \ldots, n. \tag{2.3}$$

The method of POD consists in choosing an orthonormal basis such that for every $\ell \in \{1, \ldots, d\}$ the mean square error between the elements y_j, $0 \leqslant j \leqslant 2n$, and the corresponding ℓth partial sum of (2.3) is minimized on average:

$$\begin{aligned} &\min \qquad J(\psi_1, \ldots, \psi_\ell) = \frac{1}{n} \sum_{j=1}^{n} \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle_X \psi_i \right\|_X^2 \\ &\text{subject to} \quad \langle \psi_i, \psi_j \rangle_X = \delta_{ij} \quad \text{for } 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant i. \end{aligned} \tag{2.4}$$

A solution $\{\psi_i\}_{i=1}^\ell$ to (2.4) is called *POD basis of rank* ℓ. The subspace spanned by the first ℓ POD basis functions is denoted by V^ℓ, i.e.,

$$V^\ell = \operatorname{span}\{\psi_1, \ldots, \psi_\ell\}. \tag{2.5}$$

The solution of (2.4) is characterized by the necessary optimality condition, which can be written as an eigenvalue problem. For that purpose we endow $\mathbb{R}^n$ with the weighted inner product

$$\langle v, w \rangle_{\mathbb{R}}^n = \frac{1}{n} \sum_{j=1}^{n} v_j w_j \tag{2.6}$$

for $v = (v_1, \ldots, v_n)^{\mathrm{T}}$, $w = (w_1, \ldots, w_n)^{\mathrm{T}} \in \mathbb{R}^n$ and the induced norm. Let us introduce the bounded linear operator $\mathscr{Y}_n : \mathbb{R}^n \to X$ by

$$\mathscr{Y}_n v = \frac{1}{n} \sum_{j=1}^{n} v_j y_j \quad \text{for } v \in \mathbb{R}^n. \tag{2.7}$$

Then the adjoint $\mathscr{Y}_n^* : X \to \mathbb{R}^n$ is given by

$$\mathscr{Y}_n^* z = (\langle z, y_1 \rangle_X, \ldots, \langle z, y_n \rangle_X)^{\mathrm{T}} \quad \text{for } z \in X. \tag{2.8}$$

It follows that $\mathscr{R}_n = \mathscr{Y}_n\mathscr{Y}_n^* \in \mathscr{L}(X)$ and $\mathscr{K}_n = \mathscr{Y}_n^*\mathscr{Y}_n \in \mathbb{R}^{n\times n}$ are given by

$$\mathscr{R}_n z = \frac{1}{n}\sum_{j=1}^{n}\langle z, y_j\rangle_X y_j \quad \text{for } z \in X \quad \text{and} \quad (\mathscr{K}_n)_{ij} = \frac{1}{n}\langle y_j, y_i\rangle_X, \tag{2.9}$$

respectively. The matrix $\mathscr{K}_n$ is often called a *correlation matrix*.

Using a Lagrangian framework we derive the following optimality conditions for the optimization problem (2.4):

$$\mathscr{R}_n\psi = \lambda\psi, \tag{2.10}$$

compare e.g. [6, pp. 88–91] and [47, Section 2]. Note that $\mathscr{R}_n$ is a bounded, self-adjoint and nonnegative operator. Moreover, since the image of $\mathscr{R}_n$ has finite dimension, $\mathscr{R}_n$ is also compact. By Hilbert–Schmidt theory (see e.g. [41, p. 203]) there exist an orthonormal basis $\{\psi_i\}_{i=1}^{\infty}$ for X and a sequence $\{\lambda_i\}_{i=1}^{\infty}$ of nonnegative real numbers so that

$$\mathscr{R}_n\psi_i = \lambda_i\psi_i, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_d > 0 \text{ and } \lambda_i = 0 \quad \text{for } i > d. \tag{2.11}$$

Moreover, $\mathscr{V} = \operatorname{span}\{\psi_i\}_{i=1}^{d}$. Note that $\{\lambda_i\}_{i=1}^{\infty}$ as well as $\{\psi_i\}_{i=1}^{\infty}$ depend on n. Contents permitting the notation of this dependence is dropped.

Remark 2.1. Setting

$$v_i = \frac{1}{\sqrt{\lambda_i}}\mathscr{Y}_n^*\psi_i \quad \text{for } i = 1, \ldots, d$$

we find $\mathscr{K}_n v_i = \lambda_i v_i$ and $\langle v_i, v_j \mathbb{R}^n = \delta_{ij}$ for $1 \leqslant i, j \leqslant d$. Thus, $\{v_i\}_{i=1}^{d}$ is an orthonormal basis of eigenvectors of $\mathscr{K}_n$ for the image of $\mathscr{K}_n$. Conversely, if $\{v_i\}_{i=1}^{d}$ is a given orthonormal basis for the image of $\mathscr{K}_n$, then it follows that the first d eigenfunctions of $\mathscr{R}_n$ can be determined by

$$\psi_i = \frac{1}{\sqrt{\lambda_i}}\mathscr{Y}_n v_i \quad \text{for } i = 1, \ldots, d.$$

Hence, we can determine the POD basis by solving either the eigenvalue problem for $\mathscr{R}_n$ or the one for $\mathscr{K}_n$. We will address this in Section 2.3.

The sequence $\{\psi_i\}_{i=1}^{\ell}$ solves the optimization problem (2.4). This fact as well as the error formula below were proved in [6, Section 3], for example.

Proposition 2.2. *Let* $\lambda_1 \geqslant \cdots \geqslant \lambda_d > 0$ *denote the positive eigenvalues of* $\mathscr{R}_n$ *with the associated eigenvectors* $\psi_1, \ldots, \psi_d \in X$. *Then,* $\{\psi_i^n\}_{i=1}^{\ell}$ *is a POD basis of* rank $\ell \leqslant d$, *and we have the error formula*

$$J(\psi_1, \ldots, \psi_\ell) = \frac{1}{n}\sum_{j=1}^{n}\left\| y_j - \sum_{i=1}^{\ell}\langle y_j, \psi_i\rangle_X\psi_i\right\|_X^2 = \sum_{i=\ell+1}^{d}\lambda_i. \tag{2.12}$$

2.3. POD and singular value decomposition

In this section we point out that POD is closely related to the singular value decomposition (SVD). This fact is very useful for implementation issues, in particular, for the computation of the POD basis functions ψ_i as well as of the corresponding eigenvalues λ_i. The finite-dimensional case was studied in [27].

From Remark 2.1 we infer that there exist uniquely determined real numbers $\sigma_1 \geqslant \cdots \geqslant \sigma_d > 0$ and orthonormal bases $\{v_i\}_{i=1}^n$ of $\mathbb{R}^n$ and $\{\psi_i\}_{i=1}^d$ of $\mathscr{V}$ such that

$$\mathscr{Y}_n v_i = \sigma_i \psi_i \quad \text{and} \quad \mathscr{Y}_n^* \psi_i = \sigma_i v_i \quad \text{for } i = 1, \ldots, d, \tag{2.13}$$

where $d \leqslant n$. The nonnegative numbers $\sigma_1, \ldots, \sigma_d$ are called the *singular values* of $\mathscr{Y}_n$ and (2.13) is the *singular value decomposition of* $\mathscr{Y}_n$.

Let $\tilde{\mathscr{A}} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$ and $\tilde{\mathscr{B}} = \{\tilde{\psi}_1, \ldots, \tilde{\psi}_d\}$ be two orthonormal bases of $\mathbb{R}^n$ and $\mathscr{V}$, respectively. The uniquely determined *representation matrix* $M_{\tilde{\mathscr{B}}}^{\tilde{\mathscr{A}}}(G) = ((a_{ij})) \in \mathbb{R}^{d \times n}$ of a given $G \in \mathscr{L}(\mathbb{R}^n, \mathscr{V})$ with respect to the bases $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ is defined by

$$G\tilde{v}_j = \sum_{i=1}^{d} a_{ij} \tilde{\psi}_i \quad \text{for } j = 1, \ldots, n.$$

It is well-known that the mapping $M_{\tilde{\mathscr{B}}}^{\tilde{\mathscr{A}}} : \mathscr{L}(\mathbb{R}^n, \mathscr{V}) \to \mathbb{R}^{d \times n}$ is linear and bijective.

Remark 2.3. Let $G \in \mathscr{L}(\mathbb{R}^n, \mathscr{V})$. Analogous to (2.13) there exist a uniquely determined diagonal matrix $\tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_d) \in \mathbb{R}^{d \times d}$ and orthonormal bases $\tilde{\mathscr{A}}$ of $\mathbb{R}^n$ and $\tilde{\mathscr{B}}$ of $\mathscr{V}$ so that

$$M_{\tilde{\mathscr{B}}}^{\tilde{\mathscr{A}}}(G) = (\tilde{\Sigma} \quad 0),$$

where 0 is an $d \times (n - d)$ matrix of zeros.

The previous remark motivates to the next definition.

Definition 2.4. Let G and $\tilde{\Sigma}$ be given as in Remark 2.3. We define a norm $|||\cdot|||$ on $\mathscr{L}(\mathbb{R}^n, \mathscr{V})$ by

$$|||G||| = \|\tilde{\Sigma}\|_{\mathrm{F}} = \left(\sum_{k=1}^{d} \tilde{\sigma}_k^2 \right)^{1/2} \qquad \text{for } G \in \mathscr{L}(\mathbb{R}^n, \mathscr{V}),$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Furthermore, the rank of G is given by rank $\tilde{\Sigma}$.

Remark 2.5. Since $\|\cdot\|_{\mathrm{F}}$ is a norm, it is obvious that the mapping $|||\cdot|||$ is a norm on $\mathscr{L}(\mathbb{R}^n, \mathscr{V})$. Furthermore, the properties of the Frobenius norm imply that $\|\tilde{\Sigma}\|_{\mathrm{F}} = \|M_{\mathscr{B}}^{\mathscr{A}}(G)\|_{\mathrm{F}}$ for any orthonormal bases $\mathscr{A}$ and $\mathscr{B}$ of $\mathbb{R}^n$ and $\mathscr{V}$, respectively.

Example 2.6. In many applications the snapshots $\{y_j\}_{j=1}^n$ are given by a Galerkin approach, i.e.,

$$y_j = \sum_{i=1}^k Y_{ij}\varphi_i \in X_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\} \subset X,$$

where $Y \in \mathbb{R}^{k\times n}$ and the set $\{\varphi_i\}_{i=1}^k$ is linearly independent in X_h. For $v_h = \sum_{i=1}^k v_i \varphi_i \in X_h$ and $w_h = \sum_{i=1}^k w_i\varphi_i \in X_h$ we find

$$\langle v_h, w_h\rangle_X = v^{\mathrm{T}} M w,$$

where $v = (v_1, \ldots, v_k)^{\mathrm{T}}$, $w = (w_1, \ldots, w_k)^{\mathrm{T}} \in \mathbb{R}^k$ are column vectors and

$$M = ((M_{ij})) \in \mathbb{R}^{k\times k} \quad \text{with } M_{ij} = \langle\varphi_j, \varphi_i\rangle_X$$

denotes the positive definite mass matrix. Setting $D = \operatorname{diag}(1, \ldots, 1)/\sqrt{n} \in \mathbb{R}^{n\times n}$ and $\hat{Y} = DY \in \mathbb{R}^{k\times n}$ It follows that the correlation matrix $\mathscr{K}_n = \mathscr{Y}_n^*\mathscr{Y}_n$ introduced in (2.9) is given by

$$\mathscr{K}_n = \hat{Y}^{\mathrm{T}} M \hat{Y}.$$

Now we turn to the operator $\mathscr{R}_n = \mathscr{Y}_n\mathscr{Y}_n^*$, which was also introduced in (2.9). For every $z_h \in X_h$ there exist unique coefficients $z_1, \ldots, z_k$ and $r_1, \ldots, r_k$ in $\mathbb{R}$ satisfying

$$z_h = \sum_{i=1}^k z_i\varphi_i \quad \text{and} \quad \mathscr{R}_n z_h = \sum_{i=1}^k r_i\varphi_i.$$

Then the linear mapping $\Re_n : \mathbb{R}^k \to \mathbb{R}^k$, $(z_1, \ldots, z_k) \mapsto (r_1, \ldots, r_k)$, is given by

$$\Re_n = \hat{Y}\hat{Y}^{\mathrm{T}} M.$$

Using SVD we obtain a result analogous to the one in Proposition 2.2. The proposition extends the work in [27, Section 2], where the authors studied the case, where $\mathscr{V}$ is a subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. A proof of the next result is given in [48].

Proposition 2.7 (POD and SVD). *Let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_d > 0$ denote the singular values of $\mathscr{Y}_n$ and let $\mathscr{A} = \{v_1, \ldots, v_n\}$ and $\mathscr{B} = \{\psi_1, \ldots, \psi_d\}$ be the corresponding orthonormal bases of $\mathbb{R}^n$ and $\mathscr{V}$ such that*

$$\mathscr{Y}_n v_i = \sigma_i\psi_i \quad \text{for } i = 1, \ldots, d.$$

For $\ell \leqslant d$ we define the bounded linear operator $\mathscr{Y}_n^\ell : \mathbb{R}^n \to \mathscr{V}$ by

$$\mathscr{Y}_n^\ell = \frac{1}{n}\sum_{j=1}^n \left(\sum_{i=1}^\ell \langle v, v_i\rangle_{\mathbb{R}^n} v_i\right)_j y_j \qquad \text{for } v \in \mathbb{R}^n. \tag{2.14}$$

Then the POD basis of rank ℓ is given by $\{\psi_i\}_{i=1}^\ell$. Moreover, for all $\mathscr{F}^\ell \in \mathscr{L}(\mathbb{R}^n, \mathscr{V})$ with rank $\mathscr{F}^\ell = \ell$ *we have*

$$|||\mathscr{Y}_n - \mathscr{F}^\ell|||^2 \geqslant |||\mathscr{Y}_n - \mathscr{Y}_n^\ell|||^2 = \sum_{i=\ell+1}^{d} \sigma_i^2. \tag{2.15}$$

Remark 2.8. Let us discuss the application of Proposition 2.7 to Example 2.6. The POD-basis of rank ℓ can be computed from the eigenvalue problem

$$\hat{Y}\hat{Y}^{\mathrm{T}} M\psi_i = \sigma_i^2 \psi_i \quad \text{with } \psi_i^{\mathrm{T}} M \psi_j = \delta_{ij} \text{ for } 1 \leqslant i, j \leqslant d. \tag{2.16}$$

The eigenvalue problem (2.16) can be solved by utilizing singular value analysis. Multiplying (2.16) by the positive square root $M^{1/2}$ of M from the left and setting $u_i = M^{1/2}\psi_i$ we obtain the symmetric $k \times k$-eigenvalue problem

$$\tilde{Y}\tilde{Y}^{\mathrm{T}} u_i = \sigma_i^2 u_i \quad \text{with } u_i^{\mathrm{T}} u_j = \delta_{ij} \text{ for } 1 \leqslant i, j \leqslant d, \tag{EP$_1$}$$

where $\tilde{Y} = M^{1/2}\hat{Y} = M^{1/2}DY \in \mathbb{R}^{k\times n}$. The POD basis can also be computed by using generalized singular value analysis, see e.g. [18, Theorem 8.7.4]. If we multiply (2.16) with M from the left we obtain the generalized symmetric eigenvalue problem

$$M\hat{Y}\hat{Y}^{\mathrm{T}} M\psi_i = \sigma_i^2 M\psi_i \quad \text{with } \psi_i^{\mathrm{T}} M\psi_j = \delta_{ij} \text{ for } 1 \leqslant i, j \leqslant d. \tag{EP$_2$}$$

Alternatively, we can compute the POD-basis utilizing the operator $\mathscr{K}_n$:

$$\hat{Y}^{\mathrm{T}} M\hat{Y} v_i = \sigma_i^2 v_i \quad \text{with } v_i^{\mathrm{T}} D^2 v_j = \delta_{ij} \text{ for } 1 \leqslant i, j \leqslant d, \tag{2.17}$$

where $(v_i)_j$ denotes the jth component of the eigenvector v_i. Multiplying (2.17) with D from the left and setting $w_i = Dv_i$ we obtain the symmetric $n \times n$-eigenvalue problem

$$\hat{Y}^{\mathrm{T}} M\hat{Y} w_i = \sigma_i^2 w_i \quad \text{with } w_i^{\mathrm{T}} w_j = \delta_{ij} \quad \text{for } 1 \leqslant i, j \leqslant d, \tag{EP$_3$}$$

Due to Remark 2.1 the POD basis is given by

$$\psi_i = \frac{1}{\sigma_i} Y w_i \quad \text{for } i = 1, \ldots, d.$$

For $n \ll k$ one should solve (EP$_3$) instead of (EP$_1$) or (EP$_2$). In comparison to (EP$_1$) the advantage of the generalized eigenvalue problem (EP$_2$) is the fact that the linear system $M^{1/2}\psi_i = u_i$ need not be solved.

3. ROC design problems and NSDP formulations

Depending on the ROC design goal, it is possible to derive a nonlinear semidefinite program from the specific control synthesis problem. In this section, we describe three different output feedback ROC design problems (e.g., $\mathscr{H}_2$, $\mathscr{H}_\infty$ or mixed $\mathscr{H}_2/\mathscr{H}_\infty$ design, respectively) and state the corresponding NSDPs which can be solved by IPCTR for obtaining the output feedback ROC gain of the discretized PDE system.

3.1. Output feedback reduced order control system

A typical instance of a reduced order output feedback control system can be stated as follows. Consider a (finite dimensional) linear time-invariant plant of order n_s with state space realization:

$$\begin{aligned} \dot{x}_s(t) &= A_s x_s(t) + B_s u_s(t) + B_{1s} w(t), \\ y_s(t) &= C_s x_s(t), \\ z(t) &= C_{1s} x_s(t) + D_{1s} u_s(t), \end{aligned} \tag{3.1}$$

where $x_s \in \mathbb{R}^{n_s}$, $u_s \in \mathbb{R}^{p_s}$, $y_s \in \mathbb{R}^{r_s}$, $z \in \mathbb{R}^{n_z}$, and $w \in \mathbb{R}^{n_w}$ denote the state, control input, measured output, regulated output, and noise input, respectively. We assume that the plant data $A_s, B_s, C_s, B_{1s}, C_{1s}, D_{1s}$ are appropriately dimensioned real constant matrices and the triple (A_s, B_s, C_s) is stabilizable as well as detectable (see [49]). For a given integer $0 \leqslant n_c \ll n_s$ consider the n_cth reduced order output feedback controller:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y_s(t), \\ u_s(t) &= C_c x_c(t) + D_c y_s(t), \end{aligned} \tag{3.2}$$

where $x_c \in \mathbb{R}^{n_c}$ denotes the state of the dynamic ROC law and the controller matrices $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times r_s}$, $C_c \in \mathbb{R}^{p_s \times n_c}$, $D_c \in \mathbb{R}^{p_s \times r_s}$ are not known. We collect the controller unknowns in the ROC output feedback operator F, defined by

$$F = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \in \mathbb{R}^{p \times r}, \quad p := n_c + p_s, \quad r := n_c + r_s. \tag{3.3}$$

If $n_c = 0$, the ROC system (3.2) coincides with the static output feedback (SOF) controller, e.g.,

$$u_s(t) = F y_s(t), \quad F := D_c \in \mathbb{R}^{p_s \times r_s}. \tag{3.4}$$

In this case, the ROC problem reduces to the standard SOF control design problem (see [31]). By the following well known system augmentation technique [44], it is always possible to transform the output feedback ROC problem to a SOF synthesis problem. We augment the plant state x_s by the controller state x_c and define the augmented state, control and measurement variables by

$$x(t) := \begin{bmatrix} x_s(t) \\ x_c(t) \end{bmatrix} \in \mathbb{R}^n, \quad u(t) := \begin{bmatrix} \dot{x}_c(t) \\ u_s(t) \end{bmatrix} \in \mathbb{R}^p, \quad y(t) := \begin{bmatrix} x_c(t) \\ y_s(t) \end{bmatrix} \in \mathbb{R}^r, \tag{3.5}$$

respectively, where $n := n_s + n_c$. Moreover, the augmented state space data is given by

$$\begin{aligned} &A := \begin{bmatrix} A_s & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \quad B := \begin{bmatrix} 0 & B_s \\ I_{n_c} & 0 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & I_{n_c} \\ C_s & 0 \end{bmatrix}, \\ &B_1 := \begin{bmatrix} B_{1s} \\ 0 \end{bmatrix}, \quad C_1 := \begin{bmatrix} C_{1s} & 0 \end{bmatrix}, \quad D_1 := \begin{bmatrix} 0 & D_{1s} \end{bmatrix}. \end{aligned} \tag{3.6}$$

If we replace the system quantities in (3.1) by the augmented counterparts and substitute the ROC law (3.2) into the augmented state space plant, then we get the closed loop system plant Σ_{cl} in SOF form:

$$\Sigma_{\text{cl}} \begin{cases} \dot{x}(t) = A(F)x(t) + B(F)w(t), \\ z(t) = C(F)x(t), \end{cases} \tag{3.7}$$

where $A(F) := A + BFC$, $B(F) := B_1$, $C(F) := C_1 + D_1FC$ are the augmented closed loop operators, respectively.

3.2. Constrained matrix optimization problems

We focus our discussion on three ROC design examples (ROC-$\mathscr{H}_2$, ROC-$\mathscr{H}_\infty$ and ROC-$\mathscr{H}_2/\mathscr{H}_\infty$, respectively) and the corresponding NSDP formulations. The concepts of $\mathscr{H}_\infty$ norm and $\mathscr{H}_2$ norm are well known (see [49]). Therefore, we will omit detailed discussion and content ourselves with starting the following definitions for reference. Assume the ROC law (3.2) is given such that the closed loop system (3.7) is internally stable, i.e., the real parts of the eigenvalues of $A(F)$ are all strictly negative. In this case, $A(F)$ is called *Hurwitz*. Due to the Lyapunov theorem (see [49]) there is an elegant representation of the Hurwitz property. It reduces the stability problem to an algebraic problem.

Theorem 3.1 (Lyapunov Theorem). *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{r\times n}$ be given, then the following are equivalent*:

(i) *There exists $F \in \mathbb{R}^{p\times r}$ such that $A(F) = A + BFC$ is Hurwitz.*

(ii) *For each $W \in \mathbb{R}^{n\times n}$, there exists $F \in \mathbb{R}^{p\times r}$ such that the Lyapunov equation*

$$A(F)^{\mathrm{T}}X + XA(F) + W = 0 \tag{3.8}$$

has a unique solution $X \in \mathbb{R}^{n\times n}$. If $W \succ 0$ ($W \succeq 0$), then $X \succ 0$ ($X \succeq 0$).

(iii) *There exist $F \in \mathbb{R}^{p\times r}$ and $V \in \mathscr{S}^n$ such that*

$$\mathscr{F}_s := \{F \in \mathbb{R}^{p\times r} \mid \exists V \in \mathscr{S}^n : A(F)^{\mathrm{T}}V + VA(F) = -I \prec 0, V \succ 0\} \neq \emptyset, \tag{3.9}$$

where $\mathscr{F}_s$ denotes the set of stabilizing ROC gains F.

Remark 3.2. Obviously, there exist other equivalent definitions of $\mathscr{F}_s$. For ROC design problems we prefer the definition of $\mathscr{F}_s$ as stated in (3.9). In particular, we model the internal stability of a (closed loop) system by the matrix constraints

$$\exists\, F \in \mathbb{R}^{p\times r},\ V \in \mathscr{S}^n :\ A(F)^{\mathrm{T}}V + VA(F) + I = 0, \quad V \succ 0. \tag{3.10}$$

We include the stability condition explicitly into the following matrix optimization problems as proposed by [31]. These further matrix constraints ensure the existence of a stabilizable ROC output feedback gain.

Using the stability constraints (3.10), we are able to demonstrate that the IPCTR method of Leibfritz and Mostafa [34] can be modified accordingly such that it solves the more general ROC design problems. The modification of IPCTR and the use of (3.10) is necessary for a controller of fixed order $n_c > 0$. In this case, the matrices $B_1 B_1^{\mathrm{T}}$, $C_1^{\mathrm{T}} C_1$ and $D_1^{\mathrm{T}} D_1$ will never be positive definite for the augmented matrices B_1, C_1, D_1. They are always positive semidefinite. Only in the SOF case ($n_c = 0$), one can impose some restrictions on the data matrices such as $D_1^{\mathrm{T}} D_1 \succ 0$ and $B_1 B_1^{\mathrm{T}} \succ 0$, which are common assumptions in the standard SOF/LQ literature. IPCTR in [34] also uses these assumptions for solving SOF design problems and thus, the corresponding NSDPs can use the same Lyapunov matrix for modeling the matrix equality and the stability (inequality) constraints. This is not possible for ROCs. But with the modifications described below, it is possible to adapt IPCTR such that it solves more general ROC design problems.

3.2.1. ROC-$\mathscr{H}_2$ design and NSDP formulation

The most basic optimal control problem is the following $\mathscr{H}_2$ design problem (see [31,44]):

Optimal fixed order $\mathscr{H}_2$ synthesis: *Given real matrices A, B, C, B_1, C_1, D_1 and an integer $0 \leqslant n_c < n$, find a controller gain F of order n_c such that the closed loop matrix $A(F)$ is Hurwitz and the $\mathscr{H}_2$ norm of the closed loop system* (3.7) *is minimal*.

If $A(F)$ is Hurwitz, it is well known that the $\mathscr{H}_2$ norm of Σ_{cl} is given by

$$\|\Sigma_{\mathrm{cl}}\|_{\mathscr{H}_2}^2 = \langle L, B(F)B(F)^{\mathrm{T}} \rangle = \mathrm{Tr}(L B_1 B_1^{\mathrm{T}}), \tag{3.11}$$

where $L \in \mathscr{S}^n$ satisfies the Lyapunov equation

$$A(F)^{\mathrm{T}} L + L A(F) + C(F)^{\mathrm{T}} C(F) = 0. \tag{3.12}$$

Using Theorem 3.1, (3.10) and (3.11), the optimal fixed order output feedback $\mathscr{H}_2$ problem is equivalent to the following nonlinear semidefinite program:

$$\begin{aligned} \min \quad & \mathrm{Tr}(L B_1 B_1^{\mathrm{T}}) \\ \text{s.t.} \quad & A(F)^{\mathrm{T}} L + L A(F) + C(F)^{\mathrm{T}} C(F) = 0, \\ & A(F)^{\mathrm{T}} V + V A(F) + I = 0, \ \ V \succ 0. \end{aligned} \tag{3.13}$$

For the ROC case with $n_c > 0$ it is essential to use different Lyapunov variables L and V. Here, an optimal L corresponds with the solution of the Lyapunov equation (3.12), while an optimal V together with an optimal F satisfies the stability constraints (3.10). If $n_c > 0$, then at an optimal point (F, L, V) of (3.13), we can only guarantee that, in general, $C(F)^{\mathrm{T}} C(F)$ is positive semidefinite and $L \succeq 0$. Hence, it is very likely, that the set of optimal solutions of (3.13) is empty if we use the same Lyapunov variable for the stability constraints (3.10) and the Lyapunov matrix equation (3.12). The bilinear dependence of the constraints on the free controller parameter F and the variables L, V make the problem non-convex. Moreover, the NSDP-constraints $A(F)^{\mathrm{T}} V + V A(F) = -I \prec 0$, $V \succ 0$ ensure the internal stability of the closed loop system (3.7), i. e. $A(F)$ is Hurwitz. Notice that this is the main motivation for including these constraints explicitly into (3.13) (see [31]).

3.2.2. ROC-$\mathscr{H}_\infty$ design and NSDP formulation

In a similar fashion, we are able to reformulate the optimal fixed order $\mathscr{H}_\infty$ problem as a NSDP (see [22,33,31]). $\mathscr{H}_\infty$ synthesis is an attractive model-based control design tool and it allows incorporation of model uncertainties in the control design.

Let a scalar $\gamma > 0$ be given and assume that $A(F)$ is Hurwitz and the $\mathscr{H}_\infty$ norm of (3.7) is less than γ, i. e. $\|\Sigma_{\mathrm{cl}}\|_{\mathscr{H}_\infty} < \gamma$. Then it is a standard fact (see [31, Corollary 2.1.8] or [50]) that this is equivalent to the existence of a unique matrix $L \in \mathscr{S}^n$ and $F \in \mathbb{R}^{p\times r}$ satisfying the Riccati equation

$$A(F)^{\mathrm{T}}L + LA(F) + C(F)^{\mathrm{T}}C(F) + \frac{1}{\gamma^2}LB_1B_1^{\mathrm{T}}L = 0, \tag{3.14}$$

and $A(F) + \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L$ is Hurwitz. Therefore, we formulate the optimal fixed order $\mathscr{H}_\infty$ problem as follows:

Optimal fixed order $\mathscr{H}_\infty$ synthesis: *Given real matrices A, B, C, B_1, C_1, D_1 and an integer $0 \leqslant n_c < n$, find a controller gain F of order n_c, $L \in \mathscr{S}^n$ and $\gamma > 0$ such that for minimal γ, the triple (γ, F, L) satisfies the Riccati equation (3.14) and the Hurwitz property of $A(F) + \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L$.*

Using (iii) of Theorem 3.1 and (3.14), this problem is equivalent to the non-convex and nonlinear semidefinite program:

$$\begin{aligned}
\min \quad & \gamma \\
\text{s.t.} \quad & A(F)^{\mathrm{T}}L + LA(F) + C(F)^{\mathrm{T}}C(F) + \frac{1}{\gamma^2}LB_1B_1^{\mathrm{T}}L = 0,\ \gamma > 0, \\
& \Big(A(F) + \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L\Big)^{\mathrm{T}}V + V\Big(A(F) + \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L\Big) + I = 0,\ V \succ 0.
\end{aligned} \tag{3.15}$$

In the presented form, the optimal fixed order $\mathscr{H}_\infty$-NSDP (3.15) fit into the class of NSDPs as considered in Section 4. This is not the case, if we use the strict bounded real lemma ([50]) for deriving an equivalent optimal $\mathscr{H}_\infty$-NSDP as stated in [22, Problem 2], e.g.,

$$\begin{aligned}
\min \quad & \gamma \\
\text{s.t.} \quad & X \succ 0, \quad \gamma > 0, \quad -\begin{bmatrix} A(F)^{\mathrm{T}}X + XA(F) & XB(F) & C(F)^{\mathrm{T}} \\ B(F)^{\mathrm{T}}X & -\gamma I & D(F)^{\mathrm{T}} \\ C(F) & D(F) & -\gamma I \end{bmatrix} \succ 0,
\end{aligned} \tag{3.16}$$

where, in our case, $D(F) \equiv 0$. The major drawback of (3.16) is the dimension of the bilinear matrix inequality (BMI) constraint. Moreover, due to the BMI, it is not possible to exploit the problem structure in an optimization solver. The left hand side of the BMI lies in $\mathscr{S}^{n+n_w+n_z}$. In contrast to these drawbacks, considering (3.15) instead of (3.16), the inherent problem structure of (3.15) can be completely exploited by a solver like IPCTR. This is the case, since the first order optimality conditions of (3.15) contain a set of Lyapunov equations, which can be solved successively during the computation of a Newton-like step within a conjugate gradient like procedure. Due to this, it is also not necessary to compute the Hessian of the Lagrangian in each iteration, a very time consuming process. For more details, we refer the interested reader to [31,34,35].

3.2.3. ROC-$\mathscr{H}_2/\mathscr{H}_\infty$ design and NSDP formulation

A combination of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ design objectives leads to mixed $\mathscr{H}_2/\mathscr{H}_\infty$ synthesis (see [7,26,32]). Due to the special structure of the state space plant (3.1), the regulated output of the closed loop system (3.7) is not driven (directly) by a noise input signal. Therefore, the $\mathscr{H}_2$ norm of $\Sigma_{\rm cl}$ is finite and given by (3.11), if $A(F)$ is Hurwitz. For simplifying our presentation, we have assumed (implicitly) that the representation of z and y is noise free. For the more general case, one alternative is to consider two different noise signals, e.g., w_0, w_1, and transfer functions, e.g., T_0, T_1, of the closed loop system. For example, T_0 maps w_0 into z and defines the $\mathscr{H}_2$ norm. Similarly, T_1 maps w_1 into z and is used for describing the $\mathscr{H}_\infty$ norm bound (see [7,32,26]).

The design goals are the following: For a given scalar γ, the $\mathscr{H}_\infty$ norm of the closed loop system $\Sigma_{\rm cl}$ is less than γ, $A(F)$ is Hurwitz and the $\mathscr{H}_2$ norm of $\Sigma_{\rm cl}$ is minimal. Formally, we have:

Optimal fixed order $\mathscr{H}_2/\mathscr{H}_\infty$ synthesis: *Given real matrices A, B, C, B_1, C_1, D_1, a scalar $\gamma > 0$ and an integer $0 \leqslant n_c < n$, find a controller gain F of order n_c, such that $A(F)$ is Hurwitz, $\|\Sigma_{\rm cl}\|_{\mathscr{H}_\infty} < \gamma$ and $\|\Sigma_{\rm cl}\|_{\mathscr{H}_2}$ is minimal.*

By the discussion of the $\mathscr{H}_\infty$ problem, we already know that $A(F)$ is Hurwitz and $\|\Sigma_{\rm cl}\|_{\mathscr{H}_\infty} < \gamma$ if and only if there exist F and L satisfying the Riccati equation (3.14) such that $A(F) + \frac{1}{\gamma^2} B_1 B_1^{\rm T} L$ is Hurwitz. On the other hand, the $\mathscr{H}_2$ norm of $\Sigma_{\rm cl}$ is given by (3.11) and (3.12), if $A(F)$ is Hurwitz. Obviously, a common solution which fulfills both equations may not exists. One way out of this dilemma is to introduce a further matrix variable L_0 which satisfies the Lyapunov equation and enters the objective function of the corresponding NSDP. Another possibility is due to the following Lemma, which is similar to [31, Lemma 4.1.1].

Lemma 3.3. *Consider the closed loop system $\Sigma_{\rm cl}$ and let $\gamma > 0$ be given. Suppose $A(F)$ is Hurwitz. If there exists a pair (F, L) satisfying (3.14) such that $A(F) + \frac{1}{\gamma^2} B_1 B_1^{\rm T} L$ is Hurwitz, then*

(a) $\|\Sigma_{\rm cl}\|_{\mathscr{H}_\infty} < \gamma$
(b) $0 \preceq L_0 \preceq L$, *where $L_0 \succeq 0$ satisfies (3.12). Consequently, we have*

$$\|\Sigma_{\rm cl}\|^2_{\mathscr{H}_2} = \operatorname{Tr}(L_0 B_1 B_1^{\rm T}) \leqslant \operatorname{Tr}(L B_1 B_1^{\rm T}). \tag{3.17}$$

Proof. Part (a) is clear from the discussion above (or, use [31, Corollary 2.1.8]).

(b) Substract (3.12) from (3.14) to obtain

$$A(F)^{\mathrm{T}}(L-L_0)+(L-L_0)A(F)+\gamma^{-2}\,LB_1B_1^{\mathrm{T}}L=0.$$

This is a Lyapunov equation in $(L-L_0)$. Since $A(F)$ is Hurwitz and $L_0\succeq 0$, the Lyapunov theorem implies $0\preceq L_0\preceq L$. Hence, $0\preceq B_1^{\mathrm{T}}L_0B_1\preceq B_1^{\mathrm{T}}LB_1$, which implies (3.17). □

By Lemma 3.3, the $\mathscr{H}_\infty$ constraint is automatically enforced when a solution to the Riccati equation (3.14) exists and it yields an upper bound to the $\mathscr{H}_2$ norm of Σ_{cl}. This motivates the following nonlinear semidefinite programming version of the fixed order $\mathscr{H}_2/\mathscr{H}_\infty$ problem:

$$\begin{aligned}\min\quad & \mathrm{Tr}(LB_1B_1^{\mathrm{T}})\\ \text{s.t.}\quad & A(F)^{\mathrm{T}}L+LA(F)+C(F)^{\mathrm{T}}C(F)+\frac{1}{\gamma^2}LB_1B_1^{\mathrm{T}}L=0,\\ & \Big(A(F)+\frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L\Big)^{\mathrm{T}}V+V\Big(A(F)+\frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}L\Big)=-I,\ \ V\succ 0.\end{aligned}\tag{3.18}$$

Note that we have include explicitly the last two nonlinear matrix conditions in (3.18) due to the Hurwitz assumption.

As we have seen in the previous discussion, output feedback ROC problems can be transformed into NSDPs. This is a well known fact in the control community. In example, the famous work of Boyd et al. [8] contains a huge list of full order output feedback and/or state feedback controller design examples which can be reformulated as linear (and therefore convex) semidefinite programs. In a similar fashion, most of the corresponding ROC output feedback counterparts are transformable into certain NSDPs. Unfortunately, they are in general non-convex and thus, much harder to solve. Furthermore, there is no commercial or free optimization code for the solution of NSDPs available. This is one further reason for the development of IPCTR such that it also solves NSDPs in the form of (3.13), (3.15) and (3.18), respectively.

4. Interior point constraint trust region method

To compute an output feedback gain we adapt the IPCTR algorithm of Leibfritz and Mostafa [34,35] to the broader class of NSDPs as discussed in the previous paragraph. Since a detailed description and the whole convergence analysis is far beyond the scope of this paper, in this section we state only a brief sketch of IPCTR for solving nonlinear semidefinite programs of the form

$$\begin{aligned}\min_{F,L,V}\quad & J(F,L,V)\\ \text{s.t.}\quad & H(F,L)=0,\quad G(F,L,V)=0,\quad Y(F,L,V)\succeq 0,\end{aligned}\tag{4.1}$$

where $J : \mathbb{R}^{p\times r} \times \mathscr{S}^n \times \mathscr{S}^n \to \mathbb{R}$, $H : \mathbb{R}^{p\times r} \times \mathscr{S}^n \to \mathscr{S}^n$, $G, Y : \mathbb{R}^{p\times r} \times \mathscr{S}^n \times \mathscr{S}^n \to \mathscr{S}^n$ are assumed to be sufficiently smooth. Obviously, the NSPDs (3.13), (3.15) and (3.18) of the previous section are in the form of the more general NSDP (4.1). The goal is to describe an iterative interior point approach for the solution of such NSDPs by using the same ideas and algorithmic concepts as proposed in [34] for a more special NSDP than (4.1). Note that the class of NSDPs considered in [34] is contained in the formulation of (4.1). In our presentation of IPCTR for (4.1) we follow closely Leibfritz and Mostafa [34,35] and specify only the main changes in IPCTR which are needed for the adaption. For more details we refer the interested reader to [34,35].

The main difference to the NSDP considered in [34] is that (4.1) contains a further matrix equation, e.g., $G(F, L, V) = 0$, and a further matrix variable, e.g., V. Notice that we assume explicitly, that the matrix function $H(\cdot)$ does not depend on the matrix variable V. This will simplify the presentation of the algorithm. On the other hand, this assumption is satisfied for the NSDPs presented in Section 3. Thus, this is no restriction for the problems we have mind.

The structure of the matrix constraints $H(F, L) = 0$, $G(F, L, V) = 0$ allows an explicit composition of the optimization variables (F, L, V) into variables in $\mathscr{S}^n$, e.g., (L, V), and variables in $\mathbb{R}^{p\times r}$, e.g., F. This structure is analogous to the one exhibited by many discretized optimal control problems. In the language of those problems F represents the controls, L, V represent the states, $H(F, L) = 0$, $G(F, L, V) = 0$ represent state equations, and the nonlinear SDP-constraints $Y(F, L, V) \succeq 0$ can be interpreted as state and/or control constraints.

During the whole section we use the following basic assumptions and definitions.

Assumption 4.1

(i) $\mathscr{X} := \mathbb{R}^{p\times r} \times \mathscr{S}^n \times \mathscr{S}^n$, $X := (F, L, V) \in \mathscr{X}$ and $p, r < n$.

(ii) $J : \mathscr{X} \to \mathbb{R}$, $H, G, Y : \mathscr{X} \to \mathscr{S}^n$ are at twice continuously (Frechét-) differentiable matrix functions and the mapping $H(\cdot)$ is only a function in the variables (F, L), i.e., $H_{\mathrm{V}}(X) \equiv 0$.

(iii) For given $X = (F, L, V) \in \mathscr{F}_s$, the mappings $H_{\mathrm{L}}(X)$ and $G_{\mathrm{V}}(X)$ are invertible, where $\mathscr{F}_s := \{X \in \mathscr{X} \mid Y(X) \succ 0\}$.

(iv) There exist strict feasible points $X_0 = (F_0, L_0, V_0)$ in $\mathscr{F}_s$.

Following the strategy of interior point and SDP-methods we associate with (4.1) the following barrier problem in the matrix variables $X = (F, L, V)$

$$\begin{aligned} \min \quad & \Phi^{\mu}(X) = J(X) - \mu \log\det(Y(X)) \\ \text{s.t.} \quad & H(X) = 0, \quad G(X) = 0, \end{aligned} \tag{4.2}$$

where $\mu > 0$ and $Y(X)$ is (implicitly) assumed to be positive definite. The Lagrangian function associated with the barrier problem (4.2) is defined by

$$\ell^{\mu}(X, K) = \Phi^{\mu}(X) + \langle K_h, H(X)\rangle + \langle K_g, G(X)\rangle \tag{4.3}$$

where $K := (K_g, K_h) \in \mathscr{S}^n \times \mathscr{S}^n$ are Lagrange multipliers for the equality constraints.

4.1. Structure of the NSDP problem

Since $H_{\mathrm{V}}(X) = 0$, the linearized equality constraints of (4.2) are given by

$$\begin{aligned} H^{\mathrm{lin}} &= H_{\mathrm{F}}(X)\Delta F + H_{\mathrm{L}}(X)\Delta L + H(X) = 0, \\ G^{\mathrm{lin}} &= G_{\mathrm{F}}(X)\Delta F + G_{\mathrm{L}}(X)\Delta L + G_{\mathrm{V}}(X)\Delta V + G(X) = 0, \end{aligned} \tag{4.4}$$

where $\Delta X = (\Delta F, \Delta L, \Delta V) \in \mathscr{X}$. Since the mappings $H_{\mathrm{L}}(\cdot)$, $G_{\mathrm{V}}(\cdot)$ are assumed to be invertible, (4.4) implies the following natural representation of the step

$$\Delta X = (\Delta F, \Delta L, \Delta V) = T(X)\Delta F + N(X) \tag{4.5}$$

for given $X = (F, L, V)$, where $T(X)\Delta F$ denotes the tangential and $N(X)$ the normal component of ΔX. Let $\mathscr{I} : \mathbb{R}^{p\times r} \to \mathbb{R}^{p\times r}$ be the identity mapping, the linear operator

$$\begin{aligned} &T(\cdot) = (\mathscr{I}, T_1(\cdot), T_2(\cdot)) \in \mathscr{L}(\mathbb{R}^{p\times r}, \mathbb{R}^{p\times r} \times \mathscr{S}^n \times \mathscr{S}^n), \\ &T_1(\cdot) := -H_{\mathrm{L}}^{-1}(\cdot)H_{\mathrm{F}}(\cdot),\ T_2(\cdot) := -G_{\mathrm{V}}^{-1}(\cdot)(G_{\mathrm{F}}(\cdot) - G_{\mathrm{L}}(\cdot)H_{\mathrm{L}}^{-1}(\cdot)H_{\mathrm{F}}(\cdot)), \end{aligned} \tag{4.6}$$

characterizes the null space of $(H'(\cdot), G'(\cdot))$, given by $\mathscr{N}(H', G') = \{T(X)\Delta F, \Delta F \in \mathbb{R}^{p\times r}\}$. Moreover, denoting by 0 the zero matrix, the normal component is defined by

$$\begin{aligned} N(\cdot) = \big(0, -H_{\mathrm{L}}^{-1}(\cdot)H(\cdot), \\ - G_{\mathrm{V}}^{-1}(\cdot)(G(\cdot) - G_{\mathrm{L}}(\cdot)H_{\mathrm{L}}^{-1}(\cdot)H(\cdot))\big) \in \mathbb{R}^{p\times r} \times \mathscr{S}^n \times \mathscr{S}^n. \end{aligned} \tag{4.7}$$

This problem structure is very similar to the reduced SQP structure of IPCTR as discussed in [34]. Therefore, by taking the problem structure of (4.1) into account, we can also use IPCTR as an optimization solver for (4.1). Basically, IPCTR consists of the solution of a sequence of equality constraint barrier problems of the form (4.2). It (approximately) solves (4.2) for a sequence of barrier parameters $\mu_j > 0$, $j = 0, 1, 2, \ldots$, whose limiting value is zero. Fixing $\mu_j > 0$, each barrier problem (4.2) can be solved by a particular trust region type method, the so-called tangent space approach. This approach was suggested by Byrd [9] and Omojokun [38] and has been used by many authors (see, e.g., [10,12,34]). In this approach, the trial step $\Delta X = (\Delta F, \Delta L, \Delta V)$ is decomposed into the tangential and the normal component as mentioned in (4.5). Before we specify some more details of the trust region method in IPCTR, we start by describing the general framework of the interior point method.

4.2. Sketch of the interior point algorithm

To distinguish the overall algorithm from the *inner minimization/iteration*, that is the approximate solution of the barrier subproblem (4.2), we call the former the *outer minimization/iteration*. We formally state the outer minimization as Algorithm 4.1. For a convergence analysis and more specific details of the following algorithm we refer to [34].

Algorithm 4.1. (Outer minimization/iteration, $j \geqslant 0$)

Initialization An initial barrier parameter $\mu_0 > 0$, the inner termination criterion $\epsilon_j > 0$, and $X_0 = (F_0, L_0, V_0)$ with $Y(X_0) \succ 0$ are given. Set $k = 0$.

Inner minimization/iteration Approximately solve the barrier problem (4.2). Stop as soon as an inner iterate $X_k = (F_k, L_k, V_k)$ satisfies

$$\|\nabla \ell_{\mathrm{F}}^{\mu_j}(X_k, K_k)\| + \|H(X_k)\| + \|G(X_k)\| \leqslant \epsilon_j,$$

where the multipliers $K_k := (K_h, K_g)_k$ are the solutions of the adjoint (multiplier) equations $\nabla \ell_{\mathrm{L}}^{\mu}(X, K_h, K_g) = 0$, $\nabla \ell_{\mathrm{V}}^{\mu}(X, K_h, K_g) = 0$; e.g.,

$$K_g = -(G_{\mathrm{V}}^{-1}(\cdot))^* \nabla \Phi_{\mathrm{V}}^{\mu}(\cdot), \quad K_h = -(H_{\mathrm{L}}^{-1}(\cdot))^* (\nabla \Phi_{\mathrm{L}}^{\mu}(\cdot) - G_{\mathrm{L}}^*(\cdot) K_g). \tag{4.8}$$

Choose $\mu_{j+1} < \mu_j$, $\epsilon_{j+1} < \epsilon_j$, increment j by one, and perform the next inner minimization.

Notice that in a practical implementation we also compute a predictor step for each new barrier parameter. This ensures, that the new initial guess to the next barrier problem is closer to the approximate solution of the new problem. As mentioned above, IPCTR computes an approximate solution to the barrier problem by a constrained trust region method. In the following paragraph we state some details of this algorithm applied to our NSDP.

4.3. Trust region method

We specify some details of the trust region method which we use to perform the inner iteration of Algorithm 4.1. During the whole subsection, we assume that $\mu > 0$ is fixed and a strictly feasible pair $X = (F, L, V) \in \mathscr{F}_s$ be known. The goal is to solve the barrier subproblem efficiently. As a first result of (4.5), we have the following lemma which is similar to [34, Lemma 2.2].

Lemma 4.1. *Let $X = (F, L, V) \in \mathscr{F}_s$ be given, $H_{\mathrm{V}}(\cdot) = 0$ and $H_{\mathrm{L}}(\cdot)$, $G_{\mathrm{V}}(\cdot)$ be invertible, then the linearized constraints (4.4) decompose into the following four linear equations:*

$$H_{\mathrm{L}}(X)\Delta L^n + H(X) = 0, \tag{4.9}$$

$$H_{\mathrm{L}}(X)\Delta L^t + H_{\mathrm{F}}(X)\Delta F = 0, \tag{4.10}$$

$$G_{\mathrm{V}}(X)\Delta V^n + G_{\mathrm{L}}(X)\Delta L^n + G(X) = 0, \tag{4.11}$$

$$G_{\mathrm{V}}(X)\Delta V^t + G_{\mathrm{L}}(X)\Delta L^t + G_{\mathrm{F}}(X)\Delta F = 0. \tag{4.12}$$

With Lemma 4.1 we are able to state an approximate solution of the quasi-normal problem. It is a trust region subproblem of the following form:

$$\begin{array}{ll} \min & \frac{1}{2}(\|H_{\mathrm{L}}(\cdot)\Delta\hat{L}^n + H(\cdot)\|^2 + \|G_{\mathrm{V}}(\cdot)\Delta\hat{V}^n + G_{\mathrm{L}}(\cdot)\Delta\hat{L}^n + G(\cdot)\|^2) \\ \text{s.t.} & \|(\Delta\hat{L}^n, \Delta\hat{V}^n)\| \leqslant \omega\delta, \end{array}$$

where $\omega \in (0, 1)$ is a given scalar and $\delta > 0$ denotes the current trust region radius. One approach to solve the quasi-normal problem is the following: First, compute $(\Delta\hat{L}^n, \Delta\hat{V}^n)$ by solving the linear equations (4.9), (4.11) and second, control the size of $(\Delta\hat{L}^n, \Delta\hat{V}^n)$ such that they are inside the trust region. In particular, compute a scalar $\beta \in (0, 1]$ by

$$\beta = \begin{cases} 1, & \text{if } \|(\Delta\hat{L}^n, \Delta\hat{V}^n)\| \leqslant \omega\delta, \\ \frac{\omega\delta}{\|(\Delta\hat{L}^n, \Delta\hat{V}^n)\|}, & \text{else.} \end{cases} \tag{4.13}$$

and set $(\Delta L^n, \Delta V^n) = \beta(\Delta\hat{L}^n, \Delta\hat{V}^n)$.

As a result of Lemma 4.1 and the representation of the step (4.5), the tangential component depends on ΔF and, thus, if ΔF is known, $(\Delta L^t, \Delta V^t) = (T_1(\cdot)\Delta F, T_2(\cdot)\Delta F)$ can be obtained as the solutions of the linear equations (4.10), (4.12), respectively. Therefore, the tangential problem can be reformulated as a quadratic trust region subproblem which depends only on ΔF. Particularly, a routine calculation shows, that a quadratic model of the Lagrangian functional ℓ^μ can be restated as a function ψ depending only on ΔF (see [34]). Setting again $X = (F, L, V)$, we obtain the following compact form of our quadratic model

$$\psi(\Delta F) = \langle \Delta F, T^*\nabla\Phi^\mu_X + T^*\nabla^2\ell^\mu_{XX}N\rangle + \tfrac{1}{2}\langle \Delta F, T^*\nabla^2\ell^\mu_{XX}T\Delta F\rangle, \tag{4.14}$$

where $\nabla\Phi^\mu_X = (\nabla\Phi^\mu_{\mathrm{F}}(X), \nabla\Phi^\mu_{\mathrm{L}}(X), \nabla\Phi^\mu_{\mathrm{V}}(X)) \in \mathscr{X}$, $\ell^\mu = \ell^\mu(X, K_g, K_h)$, T and N denotes the reduction mapping (4.6) and the normal operator (4.7), respectively. Notice that $T^*\nabla^2\ell^\mu_{XX}T \in \mathscr{L}(\mathbb{R}^{p\times r}, \mathbb{R}^{p\times r})$ is the reduced Hessian of the barrier Lagrangian and $T^*\nabla\Phi^\mu_X$ indicates the reduced gradient of $\Phi^\mu(X)$. Thus we search an approximate solution of the tangential trust region subproblem

$$\begin{array}{ll} \min_{\Delta F} & \psi(\Delta F) \\ \text{s.t.} & \|\Delta F\| \leqslant \delta,\ Y(X + T(X)\Delta F + \Delta X^n) \succeq (1-\sigma)Y(X) \end{array} \tag{4.15}$$

for computing the F-part of the tangential component, where $\sigma \in (0, 1)$ be a given scalar, $\Delta X^n = (0, \Delta L^n, \Delta V^n)$ and $X + T\Delta F + \Delta X^n = (F + \Delta F, L + T_1\Delta F + \Delta L^n, V + T_2\Delta F + \Delta V^n)$.

Now we apply the modified conjugate gradient (CG) algorithm as described in [34, Algorithm 2.1] for finding an approximate solution of the tangential problem (4.15). The conjugate gradient approach in [34, Algorithm 2.1] has the following properties:

(1) It solves approximatively a reduced Newton-like equation in ΔF; i. e. on exit, the approximate solution ΔF of (4.15) generated by CG satisfies approximatively the equation $\frac{\partial \psi}{\partial \Delta F} = 0$, e.g. (see [34, Lemma 2.4])

$$T^*(X)\nabla^2 \ell^\mu_{XX}(\cdot)T(X)\Delta F = -T^*(X)(\nabla \Phi^\mu_X(X) + \nabla^2 \ell^\mu_{XX}(\cdot)N(X)). \quad (4.16)$$

(2) During each CG iteration, the method computes a maximal scalar $\tau > 0$, which makes sure that the inequality constraint in (4.15) is fulfilled. In particular, on exit, it is guaranteed that the step ΔF stays inside the trust region and the point $X + T(X)\Delta F + \Delta X^n$ satisfies the SDP constraint in (4.15).
(3) In every CG iteration the operator T has to be applied. In particular, for a given conjugate direction δF_{cg}, first we solve the linear equation (4.10) for ΔL^t (e.g., $\Delta L^t = T_1(\cdot)\delta F_{cg}$). Then we put the solution of (4.10) in (4.12) and solve this linear equation for ΔV^t (e.g., $\Delta V^t = T_2(\cdot)\delta F_{cg}$).
(4) There are different ways in which the CG method can terminate:
    (i) A direction of negative curvature is encountered in the CG iteration. In this case, we follow this direction until reaching the boundary of the intersection of the trust region and the SDP-constraints. Then the resulting step is returned as an approximate solution of tangential subproblem (4.15).
    (ii) The CG iterate has stepped outside of the intersection of the trust region and the SDP-constraints. In this case, we backtrack to this region and return the resulting step as an approximate solution of (4.15).
    (iii) The algorithm terminates with a pre-specified inexact termination criterion.

The advantages of this strategy are: The CG-loop works only in the space of the F-variable, which is in general much smaller as the abstract state space $\mathscr{S}^n \times \mathscr{S}^n$, where the variables (L, V) lives in. There is no need for evaluating the Hessian of the Lagrangian explicitly. We only need to evaluate it applied to a direction. On exit, it is guaranteed that the matrix inequality is strictly satisfied. For given ΔF, the L- and V-part of the tangential component ΔX^t can be obtained by solving the linear (matrix) equations (4.10) and (4.12), respectively.

Taking the modifications as discussed above into account, the constraint trust region algorithm for the solution of the barrier problem (4.2) can be formulated similar to [34, Algorithm 2.2]. Therefore, we state only an outline of [34, Algorithm 2.2] for reference.

Algorithm 4.2. (Trust region framework)

Initialization Let $0 < \gamma < 1$, $\mu > 0$, $\epsilon > 0$, $X_0 = (F_0, L_0, X_0)$ with $Y(X_0) \succ 0$ be given. Set $k = 0$. Calculate the Lagrange multiplier estimates $K_0 = (K_g, K_h)_0$ by solving the adjoint equations in (4.8), and pick $\delta_0 > 0$.

Trust region iteration until $\|\nabla \ell_{\mathrm{F}}^{\mu_j}(X_k, K_k)\| + \|H(X_k)\| + \|G(X_k)\| \leqslant \epsilon$

1. Compute the normal step $(\Delta L_k^n, \Delta V_k^n)$ and solve the tangential problem (4.15) for ΔF_k by CG algorithm (see [34, Algorithm 2.1]).
2. Determine $(\Delta L_k^t, \Delta V_k^t) = (T_1(\cdot)\Delta F_k, T_2(\cdot)\Delta F_k)$ by solving (4.10), (4.12) and set $\Delta X_k = \Delta X_k^n + \Delta X_k^t = (\Delta F_k, \Delta L_k^n + \Delta L_k^t, \Delta V_k^n + \Delta V_k^t)$.
3. Compute the multiplier estimate K_{k+1} by solving (4.8).
4. Evaluate the ratio $r_k = \mathrm{ared}(\Delta X_k)/\mathrm{pred}(\Delta X_k)$ of the actual and predicted reduction.
5. Update the penalty parameter by the scheme stated in [34, Algorithm 2.2]).
6. Update the trust region and accept or reject the trial step ΔX_k:
   (a) If the ratio is too small, shrink the trust region and reject the step, e.g., if $r_k < \gamma$, set $\delta_{k+1} = 0.2 \max\{\|\Delta X_k^n\|, \|\Delta F_k\|\}$ and $X_{k+1} = X_k$.
   (b) Otherwise, increase the trust region and accept the step, e.g., if $r_k \geqslant \gamma$, set $\delta_{k+1} = 2\delta_k$ and $X_{k+1} = X_k + \Delta X_k$.

For defining the actual problem reduction, we use the augmented Lagrangian

$$\Lambda(X, K_h, K_g) = \ell^{\mu}(X, K_h, K_g) + \rho(\|G(X)\|^2 + \|H(X)\|^2)$$

as a merit function. In particular, we define the actual and predicted model reduction by

$$\begin{aligned}\mathrm{ared}(\Delta X_k) &= \Lambda(X_k, K_{g,k}, K_{h,k}) - \Lambda(X_k + \Delta X_k, K_{g,k+1}, K_{g,k+1}),\\ \mathrm{pred}(\Delta X_k) &= \Psi_k(0) - \Psi_k(\Delta F_k) - \langle K_{g,k+1} - K_{g,k}, G_k^{\mathrm{lin}}\rangle - \langle K_{h,k+1}\\ &\qquad - K_{h,k}, H_k^{\mathrm{lin}}\rangle + \rho_k(\|G_k\|^2 - \|G_k^{\mathrm{lin}}\|^2) + \rho_k(\|H_k\|^2 - \|H_k^{\mathrm{lin}}\|^2),\end{aligned}$$

where G_k^{lin}, H_k^{lin} denotes the linearized constraints (4.4) evaluated at X_k, ΔX_k and $H_k = H(X_k)$, $G_k = G(X_k)$, respectively. For more details and some convergence results, we refer the interested reader to [12,34].

4.4. More algorithmic details of IPCTR

In this paragraph we state some specific features of the overall algorithm, the IPCTR method, which is a combination of Algorithm 4.1 and the trust region Algorithm 4.2. Especially, we use the trust region method for the inner minimization process of the interior point method. The choice of the barrier parameter sequence $\{\mu_j\}$, the forcing sequence $\{\epsilon_j\}$ and the initial guess (the predictor step) are essential for the performance of the algorithm. In an implementation of IPCTR, we have chosen the parameters μ_j and ϵ_j in the spirit of Dussault [15] and Gould et al. [19]. In particular, if the actual barrier parameter $\mu_j < 1$ we set

$$\mu_{j+1} = \Omega\Big(\mu_j^{\frac{4m}{2m+1}}\Big), \quad m \in \mathbb{N},$$

which is equivalent to $\mu_{j+1} \geqslant \mu_j^{\frac{4m}{2m+1}}$, where for related positive quantities α and β, we write $\alpha = \mathcal{O}(\beta)$ if there is a constant $\kappa > 0$ such that $\alpha \geqslant \kappa\beta$ for all β sufficiently small and $\alpha = \Omega(\beta)$ if $\beta = \mathcal{O}(\alpha)$. Otherwise, we choose $\mu_{j+1} = a\mu_j$, $a \in (0, 1)$. Using this rule, the rate at which the barrier parameter approaches zero can be made as close to quadratic as one desires.

The updating rule for the inner minimization termination criterion is given by

$$\epsilon_{j+1} = \mathcal{O}\Big(\mu_{j+1}^{\frac{1}{m}}\Big).$$

In our practical implementation, we choose $m \in \{2, 3, 4, 5\}$.

A simple choice for the starting guess to the next barrier problem for the inner minimization loop is just the approximate solution of the current barrier problem. If the barrier parameter is sufficiently small, a better choice of the initial point to the next subproblem is the following predictor strategy. We compute the initial guess X_0 in Algorithm 4.1 for μ_{j+1} by a combination of one further reduced Newton iteration with a reduced extrapolation step. In particular, we compute

$$X_0 = X_j + \Delta X^{\mathrm{ex}}(\mu_{j+1} - \mu_j) + \Delta X^{\mathrm{ne}}, \tag{4.17}$$

where X_j is the approximate solution to the current barrier problem, $\Delta X^{\mathrm{ne}} = (\Delta F^{\mathrm{ne}}, \Delta L^{\mathrm{ne}}, \Delta V^{\mathrm{ne}})$, $\Delta X^{\mathrm{ex}} = (\Delta F^{\mathrm{ex}}, \Delta L^{\mathrm{ex}}, \Delta V^{\mathrm{ex}})$. The reduced Newton step ΔX^{ne} can be computed by first solving the reduced Newton equation

$$T^*(X_j)\nabla^2 \ell^{\mu}_{XX}(\cdot)T(X_j)\Delta F^{\mathrm{ne}} = -T^*(X_j)(\nabla \Phi^{\mu}_X(X_j) + \nabla^2 \ell^{\mu}_{XX}(\cdot)N(X_j))$$

and then $\Delta L^{\mathrm{ne}} = T_1(X_j)\Delta F^{\mathrm{ne}} + \Delta L^n$, $\Delta V^{\mathrm{ne}} = T_2(X_j)\Delta F^{\mathrm{ne}} + \Delta V^n$. Similarly, we compute the reduced extrapolation step ΔX^{ex} by first solving

$$T^*(X_j)\nabla^2 \ell^{\mu}_{XX}(\cdot)T(X_j)\Delta F^{\mathrm{ex}} = -T^*(X_j)\left(\frac{\mathrm{d}}{\mathrm{d}\mu}\nabla \ell^{\mu}_X(X_j)\right)$$

and then $\Delta L^{\mathrm{ex}} = T_1(X_j)\Delta F^{\mathrm{ex}}$, $\Delta V^{\mathrm{ex}} = T_2(X_j)\Delta F^{\mathrm{ex}}$. Finally, the initial estimates for the Lagrange multipliers $K_0 = (K_h, K_g)_0$ are given as the solutions of the adjoint equations (4.8) evaluated at X_0. If the inner minimization corresponding to μ_{j+1} is started with initial guess (4.17) and assuming the subsequent reduced Newton step X_1 is acceptable to the inner minimization trust region method, then one can show (see, e.g., [15,19]):

$$\begin{aligned}\|\nabla \ell_{\mathrm{F}}^{\mu_{j+1}}(X_1, K_1)\| + \|H(X_1)\| + \|G(X_1)\| &= \mathcal{O}\big(\mu_j^4/\mu_{j+1}^2\big)\\ &\leqslant \epsilon_{j+1} = \mathcal{O}\Big(\mu_{j+1}^{\frac{1}{m}}\Big),\end{aligned}$$

which imply locally a two-step superlinear convergence rate (one step is the combined Newton/extrapolation step and the other is one subsequent Newton iteration).

5. Numerical results

As examples for an application of output feedback ROC design we consider two nonlinear parabolic control problems. For the presented nonlinear examples, we compute the linear output feedback ROC law (3.2) from the linearized control system. Instead of computing the ROC output feedback operator F (3.3) directly from the discretized (uncontrolled) nonlinear PDE problem, first, we reduce the dimension of the large dimensional PDE model by using the POD method as described in Section 2. Then we compute the ROC output feedback operator F for the corresponding linearized (uncontrolled) low dimensional POD closed loop system of the form (3.7) by solving the $\mathscr{H}_2/\mathscr{H}_\infty$-NSDP (3.18) with IPCTR (see Section 4). Alternatively we can also solve one of the other ROC-NSDPs stated in Section 3.2. Finally, we plug in the computed POD–ROC feedback gain Section 3.2. into the large dimensional nonlinear PDE model. In the numerical examples we will demonstrate that the output feedback ROC computed for the linearized POD system can be used for controlling the large dimensional nonlinear PDE control problem. All of our computations has been done by using the MATLAB environment.

5.1. Example (Perturbed nonlinear heat equation)

The first example is a case study of a two dimensional heat equation with boundary control inputs and a nonlinear radiation term on one part of the boundary. For motivation we formulate the infinite dimensional control problem.

Let $v(\xi, \eta; t)$ denote the temperature at time $t > 0$ and at $(\xi, \eta) \in \Omega \subset \mathbb{R}^2$, where Ω is an U-shaped domain and $\partial\Omega = \bigcup_{j=1}^{8} \Gamma_j$ is the boundary of Ω. In particular, for $0 < a_1 < a_2$, $0 < b_1 < b_2 < b_3$, we choose

$$
\begin{aligned}
\Gamma_1 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi = a_2, \eta \in [b_2, b_3]\},\\
\Gamma_2 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \in [0, a_2], \eta = b_3\},\\
\Gamma_3 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi = 0, \eta \in [0, b_3]\},\\
\Gamma_4 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \in [0, a_2], \eta = 0\},\\
\Gamma_5 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi = a_2, \eta \in [0, b_1]\},\\
\Gamma_6 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \in [a_1, a_2], \eta = b_1\},\\
\Gamma_7 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi = a_1, \eta \in [b_1, b_2]\},\\
\Gamma_8 &= \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \in [a_1, a_2], \eta = b_2\},
\end{aligned}
$$

and set $a_1 = \frac{1}{2}$, $a_2 = 1$, $b_1 = \frac{1}{3}$, $b_2 = \frac{2}{3}$, $b_3 = 1$. Furthermore, we use the following notation and parameters for the thermal properties of copper:

$C = 0.0914$: heat capacity,

$\lambda = 0.05$: heat conduction,

$\rho = 8.94$: density,

$\kappa = \dfrac{\lambda}{C\rho}$: diffusion coefficient,

$\alpha_4 = 0.1$: heat exchange factor on Γ_4,

$\hat{\alpha} = 0.2$: heat exchange factor on Γ_j, $j = 6, 7, 8$,

$v_4^a = 1700$: ambient temperature on Γ_4,

$\hat{v}^a = 400$: ambient temperature on Γ_j, $j = 6, 7, 8$,

u_4: boundary control on Γ_4,

$\hat{u}$: boundary control on Γ_j, $j = 6, 7, 8$,

$\varepsilon_4 = 0.00023$: radiation coefficient on Γ_4,

$\sigma = 5.6697 \times 10^{-8}$: Stefan–Boltzmann constant

v: temperature in K on Ω,

$v_0 = 850$ K: initial temperature on Ω,

y_i: observation information to the control variables at time $t > 0$ $(i = 1, 2, 3)$.

The two dimensional heat equation with boundary output feedback control inputs and a nonlinear radiation term is given as follows: All $(v(\xi, \eta; t), u_4(t), \hat{u}(t))$ satisfy the perturbed diffusion equation

$$v_t(\xi, \eta; t) = \kappa(\Delta + \delta)v(\xi, \eta; t), \quad \text{in } \Omega, t > 0, \tag{5.1}$$

where δ denotes a linear perturbation operator, with boundary and initial conditions

$$\begin{aligned}
&-\lambda \tfrac{\partial v}{\partial n}(\xi, \eta; t) = 0, \quad \text{on } \Gamma_j, \quad j = 1, 2, 3, 5, t > 0,\\
&-\lambda \tfrac{\partial v}{\partial n}(\xi, \eta; t) = \alpha_4(v(\xi, \eta; t) - v_4^a + u_4(t))\\
&\qquad\qquad + \varepsilon_4\sigma(v(\xi, \eta; t)^4 - (v_4^a)^4), \quad \text{on } \Gamma_4, t > 0, \qquad (5.2)\\
&-\lambda \tfrac{\partial v}{\partial n}(\xi, \eta; t) = \hat{\alpha}(v(\xi, \eta; t) - \hat{v}^a + \hat{u}(t)), \quad \text{on } \Gamma_j, j = 6, 7, 8, t > 0,\\
&v(\xi, \eta, 0) = v_0(\xi, \eta), \quad \text{in } \Omega.
\end{aligned}$$

Thus, the system is described by a linear perturbed partial differential equation (the heat equation) coupled, through the boundary conditions to a nonlinear radiation term. We are interested in using sensed information to design a output feedback ROC law of the form (3.2). In particular, we assume that there are only two control inputs acting exclusively on the boundary parts Γ_4 and $\Gamma_6 \cup \Gamma_7 \cup \Gamma_8$, and the only measured information available to these controls is the temperature at time t at three fixed sensor locations on the boundary of the domain Ω. In particular, there are three observations y_i, $i = 1, 2, 3$, where

$$y_1(t) = v(0, b_3; t), \quad y_2(t) = v(0, 0; t), \quad y_3(t) = v(a_1, \tfrac{1}{2}b_3; t). \tag{5.3}$$

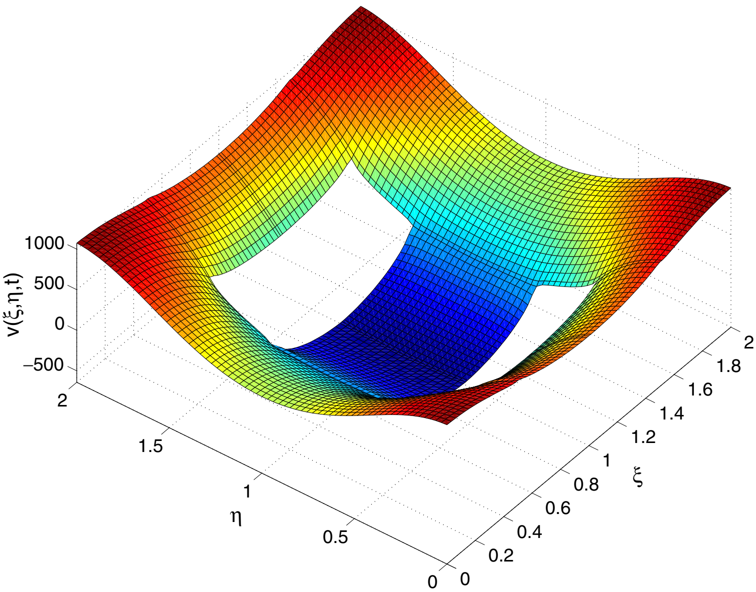

Fig. 5.1. Controlled case at $T = 20$.

It is well known that the system governed by Eqs. (5.1)–(5.3) can be written as an abstract dynamical system of the form (2.1) in an appropriate (infinite dimensional) state space. For computing the linear output feedback ROC law of the form $u(t) = Fy(t)$, $F \in \mathbb{R}^{p\times r}$, we apply standard finite difference approximation techniques to the infinite dimensional system (5.1)–(5.3). Although it is not essential to use finite differences, we restrict our attention to this approximation approach because it is easy to implement. Alternatively, a finite element approximation can also be used. We discretize the domain Ω by a uniform grid, where $h = \frac{a_2}{66} \approx 0.01515$ is the spatial step size in ξ- and η-direction. The resulting number of grid points is $n = 3796$. Thus, we obtain the following large scale, finite dimensional nonlinear approximation of the infinite dimensional control system (5.1)–(5.3):

$$\begin{aligned} &E\dot{x}(t) = (A + \delta A)x(t) + G(x(t)) + Bu(t) + B_1 w(t), \quad x(0) = x_0, \\ &y(t) = Cx(t), \end{aligned} \tag{5.4}$$

with only two control inputs u ($p = 2$) and three measured output variables y ($r = 3$). Notice that $E \in \mathbb{R}^{n\times n}$ denotes a regular diagonal matrix. Due to the fixed ambient temperature $v^a = [v_4^a, \hat{v}^a]^{\mathrm{T}}$ on some parts of the boundary, the term $B_1 w(t)$ with $w(t) \equiv [1, \ldots, 1] \in \mathbb{R}^{n_w}$ represents this constant part in our discrete model. The matrix $\delta A \in \mathbb{R}^{n\times n}$ approximates the perturbation operator δ and

$$G : \mathbb{R}^n \to \mathbb{R}^n, \quad G(x(t)) := Nx(t)^4, \quad N \in \mathbb{R}^{n\times n}$$

models the approximation of the nonlinear boundary part $\varepsilon\sigma v(\cdot; t)^4$ on Γ_4. Note that we have chosen $\delta A = 0.3825 \cdot I$ such that the real part of the largest eigenvalue

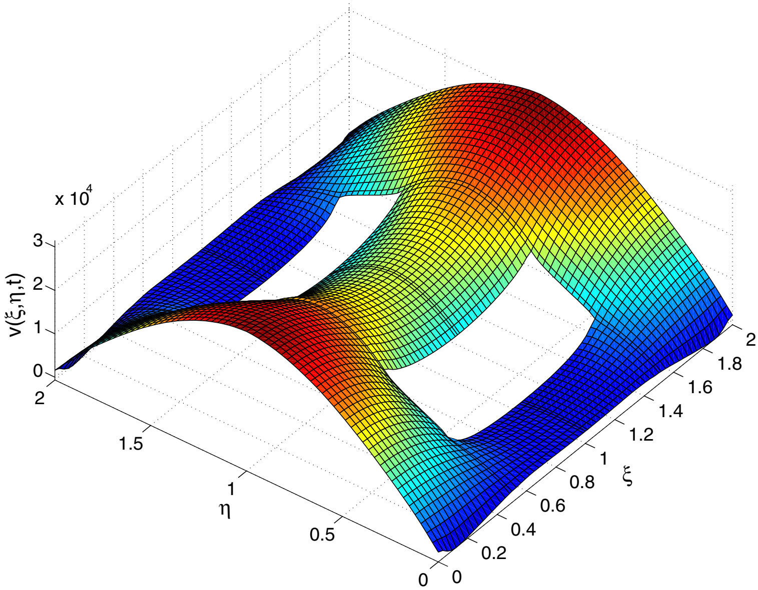

Fig. 5.2. Uncontrolled case at $T = 20$.

of $(A + \delta A)$ is positive, i. e. the perturbed (nonlinear) model (5.4) is unstable. Premultiplying the nonlinear ODE in (5.4) by E^{-1} and neglecting the nonlinear term $E^{-1}G(x(t))$, leads to the corresponding (augmented) linear control system (3.1) with data matrices (3.6) if we redefine $(A + \delta A)$, B, B_1 by $A := E^{-1}(A + \delta A)$, $B := E^{-1}B$, $B_1 := E^{-1}B_1$. For computing an output feedback ROC gain F of this unstable (linear) system, we must solve a large scale nonlinear and non-convex matrix optimization problem as discussed in Section 3.2. In particular, the total number of decision variables of the $\mathscr{H}_2/\mathscr{H}_\infty$-NSDP (3.18) is given by

$$n(n+1) + pr = 14.413418 \times 10^6, \quad (n = 3796,\ p = 2,\ r = 3).$$

As far as we know, solving a NSDP with more than 14 million variables is impossible. Therefore, we compute only five POD basis functions from the snapshots of the uncontrolled nonlinear system (for $u = 0$) which results in a very low dimensional POD system of order $n_{\text{pod}} = 5$. Then, in a second step, we solve the corresponding low dimensional $\mathscr{H}_2/\mathscr{H}_\infty$-NSDP (3.18) for computing the ROC gain F (with $n_c = 0$) of the linearized (unstable) POD system (e.g., we have deleted the nonlinear term). Note that the total number of variables of the POD–NSDP is now given by

$$n_{\text{pod}}(n_{\text{pod}} + 1) + pr = 36, \quad (n_{\text{pod}} = 5,\ p = 2,\ r = 3).$$

The plots in Figs. 5.1 and 5.3 illustrates pretty well that the output feedback ROC law of order $n_c = 0$ protect the material (here: copper) from overheating (melting

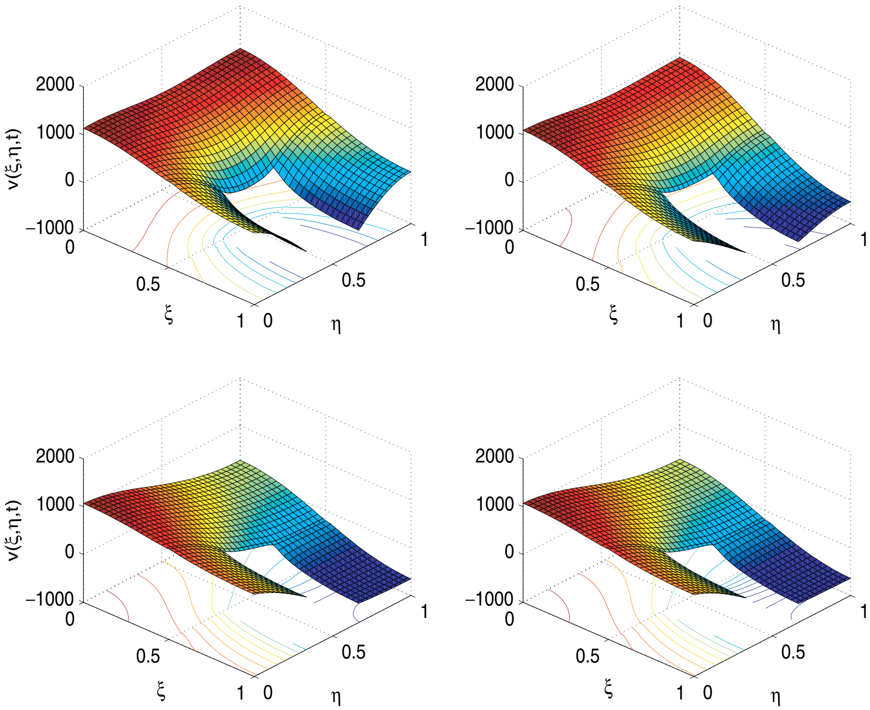

Fig. 5.3. Controlled case at $t = 1.5, 3, 10, 20$.

temperature: 1356.2 K). The hot spot of the material is 300 K below the melting temperature of copper. Notice that the inner holes in the domain of Ω can be interpreted as cooling channels which are embedded in the material. Using the POD–ROC law we achieve the stable stationary temperature distribution of copper after 20 s. For discretizing the time axis, we have chosen a uniform time grid with step size $\mathrm{d}t = 0.08333$. On the other hand, the graphs in Figs. 5.2 and 5.4 visualizes the instability of the uncontrolled nonlinear system. In the uncontrolled case, the temperature increases rapidly. After 1.5 s of the heating process, the temperature is above the melting temperature of copper. Roughly speaking, without any control, the system burns out completely.

The optimal output feedback ROC boundary control input (computed by IPCTR) can be found in Fig. 5.5. The blue dash dotted curve represents the control input $\hat{u}$ acting on the boundaries $\Gamma_6 \cup \Gamma_7 \cup \Gamma_8$. Or, in other words, $\hat{u}$ controls the cooling action of the system in the cooling channels. On the other hand, the red solid line represents the feedback control input u_4 on Γ_4. This control acts on the outer boundary of the domain, where the ambient temperature $v_4^a = 1700$ K is larger then the melting temperature of copper.

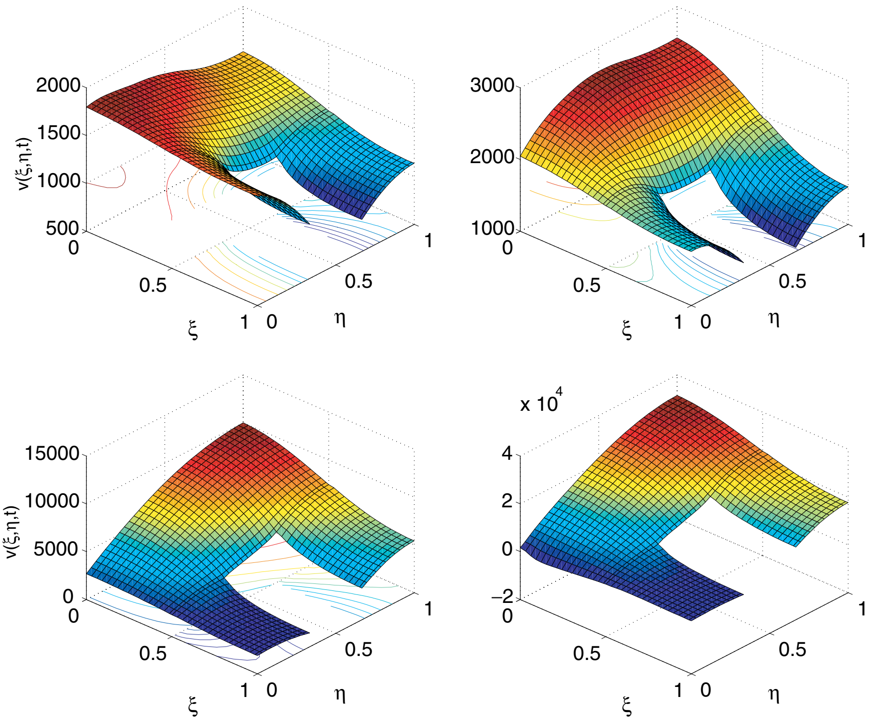

Fig. 5.4. Uncontrolled case at $t = 1.5, 3, 10, 20$.

The convergence behavior of IPCTR for computing the ROC output feedback gain F for the low dimensional POD control system can be found in Fig. 5.6. For this instance, IPCTR computes the solution of the corresponding nonlinear semidefinite program within 1.372 CPU seconds on a DELL notebook with a Pentium III, 1.0 GHz, processor. It needs 8 outer and a total of 82 inner iterations for determining the optimal ROC output feedback gain F. This figure also demonstrates that the algorithm can achieve global linear as well as fast local convergence rates. In particular, for a sufficiently small barrier parameter, the local convergence rate is superlinear (see the last two iteration dots in Fig. 5.6).

Summing up, this simple output feedback ROC law with only two control inputs and three observation points is able to stabilize the nonlinear and unstable large dimensional control system quite well. This example illustrates that the combination of a POD model reduction and nonlinear semidefinite programming can be considered as a useful tool for the design of reduced order output feedback control laws for nonlinear and unstable PDE models.

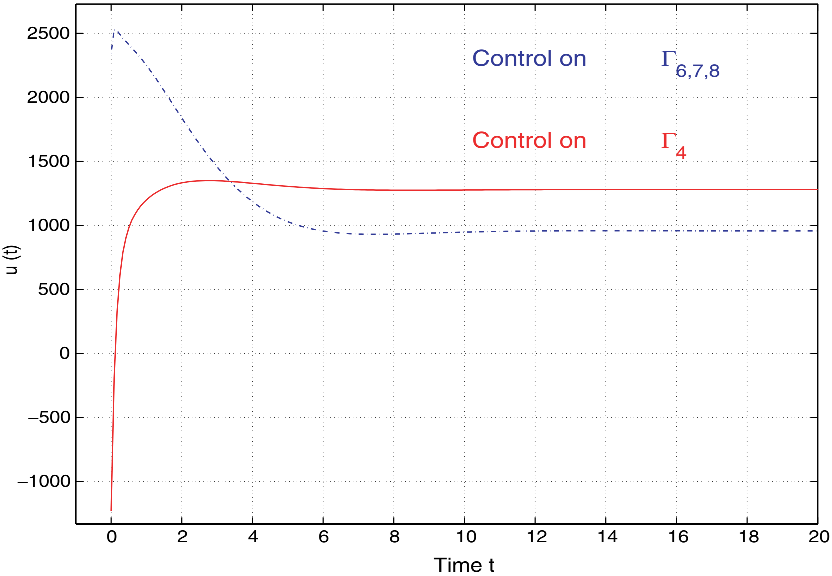

Fig. 5.5. Optimal output feedback ROC $u = Fy$.

5.2. Example (Viscous Burgers' equation)

In this example we consider the viscous Burgers equation

$$v_t - \nu v_{\xi\xi} + v v_\xi = 0 \quad \text{in } Q = (0, T) \times \Omega, \tag{5.5a}$$

$$\nu v'(t, 0) = 0 \quad \text{for all } t \in (0, T), \tag{5.5b}$$

$$\nu v'(t, 2\pi) = u \quad \text{for all } t \in (0, T), \tag{5.5c}$$

$$v(0, \xi) = \sin(\xi) \quad \text{for all } \xi \in \Omega, \tag{5.5d}$$

where $\nu = 0.05$ denotes a viscosity parameter, $T > 0$ is the end time, $u \in L^2(0, T)$ is the control input and $\Omega = (0, 2\pi)$. There are only one control input acting on the right-end of the interval Ω. The only measured information available to this control is the state at time $t \in (0, T)$ at $\xi = 2\pi$, i.e.,

$$y(t) = v(t, 2\pi) \quad \text{for all } t \in (0, T). \tag{5.5e}$$

As in the previous example, we can express (5.5a) by a dynamical system in an appropriate (infinite-dimensional) state space. The goal of our optimal control problem is to compute an output feedback ROC law as in (3.2) to track the system to zero, i.e., to minimize the cost

$$J(v, u) = \frac{1}{2} \int_0^{\mathrm{T}} \int_\Omega |v(t, \xi)|^2 + |v_\xi(t, \xi)|^2 \mathrm{d}\xi \mathrm{d}t + \frac{\beta}{2} \int_0^{\mathrm{T}} |u(t)|^2 \mathrm{d}t,$$

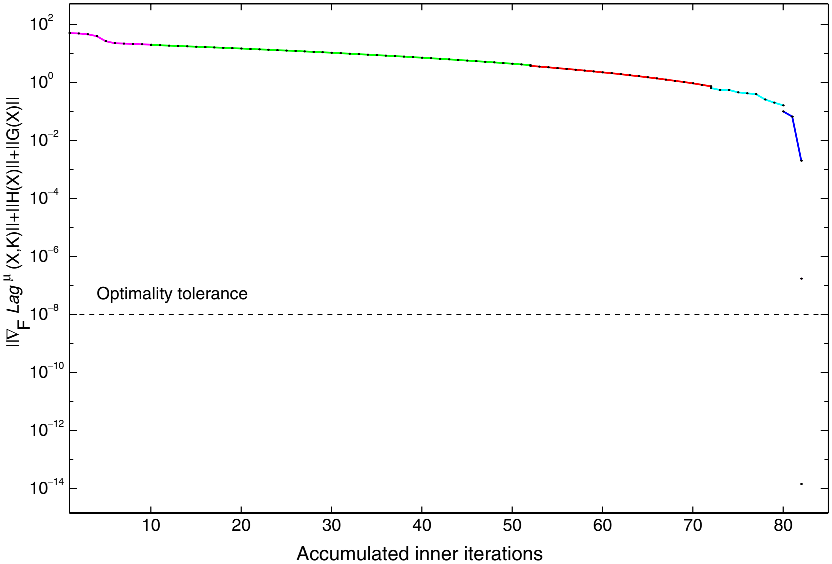

Fig. 5.6. Convergence: IPCTR.

where $\beta = 0.0005$ is a fixed regularization parameter. As in Example 5.1 we compute the ROC output feedback gain F with $n_c = 0$ for the linearized control system, i. e. we delete the nonlinear term vv_x in (5.5a). For the finite element discretization we utilize the software Femlab, Version 2.2, where we took quadratic Lagrange elements with 2515 degrees of freedom. The uncontrolled solution is presented in the left plot of Fig. 5.7 while the controlled case is illustrated in the right graph of the same figure.

In Fig. 5.8 the mapping $t \mapsto |v(t, 2\pi)|$ is plotted. For the computation of the POD basis we utilize $X = L^2(\Omega)$ and $X = H^1(\Omega)$. The computation of the POD basis for $\ell \leqslant 6$ consumes less than 2 s CPU time. Here, we determine the eigenvalues and

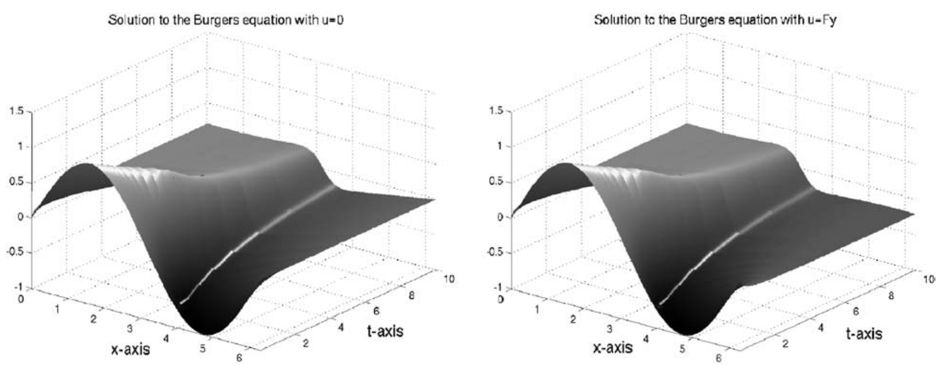

Fig. 5.7. Uncontrolled (left) and controlled (right) solution to the Burgers equation.

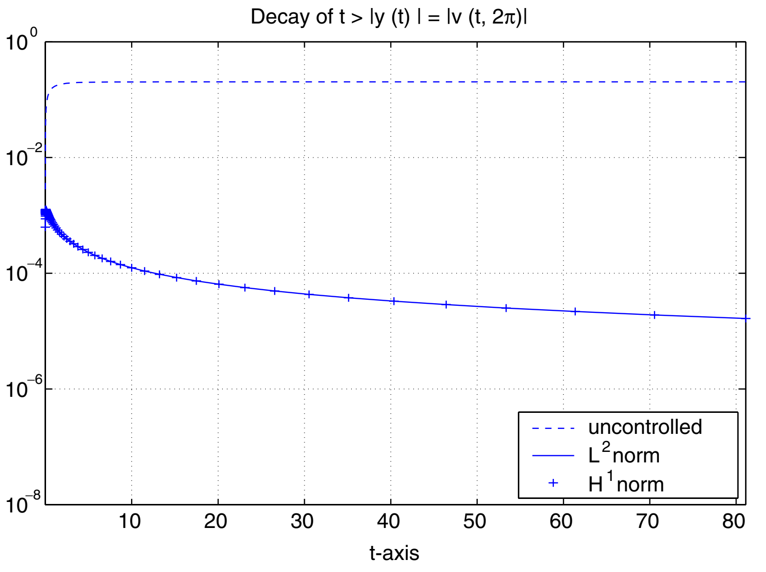

Fig. 5.8. Decay rate: absolute value at $\xi = 2\pi$.

eigenvectors of the correlation matrix $\mathscr{K}_n$ by calling the Matlab routine `eigs` and obtain the POD basis as explained in Remark 2.1. In particular, due to the measurements at $x = 2\pi$, the values of $|v(t, 2\pi)|$ decay to values that are smaller than 10^{-4} whereas the values $|v(t, 2\pi)|$ of the uncontrolled dynamics achieve a constant value close to 0.1, see Fig. 5.8. The results do not change very much if we take $\ell = 6$ POD basis function. For this example, the IPCTR algorithm consumes less than 2 s CPU time and requires 15 total inner and five outer iterations.

References

[1] K. Afanasiev, M. Hinze, Adaptive control of a wake flow using proper orthogonal decomposition, in: J. Cagnol (Ed.), et al., in: Shape optimization and optimal design. Proceedings of the IFIP Conference. Selected papers from the sessions Distributed parameter systems and Optimization methods and engineering design held within the 19th conference on system modeling and optimization, Cambridge, GB, July 12–16, 1999, Lect. Notes Pure Appl. Math., vol. 216, Marcel Dekker, New York, NY, 2001, pp. 317–332.

[2] F. Alizadeh, J.-P.A. Haeberly, M.L. Overton, Primal–dual interior point methods for semidefinite programming: convergence rates, stability and numerical results, SIAM J. Optim. 8 (1998) 746–768.

[3] E. Arian, M. Fahl, E.W. Sachs, Trust-region proper orthogonal decomposition for flow control, Technical Report 2000-25, ICASE, 2000.

[4] J.A. Atwell, J.T. Borggaard, B.B. King, Reduced order controllers for Burgers' equation with a nonlinear observer, Int. J. Appl. Math. Comput. Sci. 11 (6) (2001) 1311–1330.

[5] H.T. Banks, M.L. Joyner, B. Winchesky, W.P. Winfree, Nondestructive evaluation using a reduced-order computational methodology, Inverse Problems 16 (2000) 1–17.

[6] G. Berkooz, P. Holmes, J.L. Lumley, Turbulence, Coherent Structures, Dynamical Systems and Symmetry. Cambridge Monographs on Mechanics, Cambridge University Press, NY, 1996.
[7] D.S. Bernstein, W.M. Haddad, LQG control with an $\mathscr{H}_\infty$ performance bound: a Riccati equation approach, IEEE Trans. Automat. Control 34 (1989) 293–305.
[8] S.P. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1994.
[9] R.H. Byrd, Robust trust region methods for nonlinearly constrained optimization presented at SIAM Conference on Optimization, Houston, 1987.
[10] A.R. Conn, N.I.M. Gould, P. Toint, Trust-region methods, MPS–SIAM Series on Optimization, SIAM publications, Philadelphia, PA, USA, 2000.
[11] R.F. Curtain, H.J. Zwart, An Introduction to Infinite Dimensional Linear System Theory, Texts in Applied Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, 1995.
[12] J.E. Dennis, M. Heinkenschloss, L.N. Vicente, Trust-region interior-point SQP algorithms for a class of nonlinear programming problems, SIAM J. Control Optim. 36 (1998) 1750–1794.
[13] J. Doyle, K. Zhou, K. Glover, B. Bodenheimer, Mixed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance objectives II: Optimal control, IEEE Trans. Automat. Control 39 (1994) 1575–1587.
[14] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, State-space solutions to standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems, IEEE Trans. Automat. Control 34 (1989) 831–847.
[15] J.-P. Dussault, Numerical stability and efficiency of penalty algorithms, SIAM J. Numer. Anal. 32 (1995) 296–317.
[16] K. Fukunaga, Introduction to Statistical Recognition, Academic Press, New York, 1990.
[17] P. Gahinet, P. Apkarian, A linear matrix inequality approach to $\mathscr{H}_\infty$ control, Int. J. Robust Nonlinear Control 4 (1994) 421–448.
[18] G.H. Golub, C.F. Van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore and London, 1996.
[19] N.I.M. Gould, D. Orban, A. Sartenaer, P.L. Toint, Superlinear convergence of primal–dual interior point algorithms for nonlinear programming, SIAM J. Optim. 11 (2001) 974–1002.
[20] W.M. Haddad, V. Kapila, D.S. Bernstein, Robust $\mathscr{H}_\infty$ stabilization via parameterized Lyapunov bounds, IEEE Trans. Automat. Control 42 (1997) 41–248.
[21] D. Hömberg, S. Volkwein, Control of laser surface hardening by a reduced-order approach utilizing proper orthogonal decomposition, Math. Comput. Model. 38 (2003) 1003–1028.
[22] C.W.J. Hol, C.W. Scherer, E.G. van der Meché, O.H. Bosgra, A nonlinear SDP approach to fixed-order controller synthesis and comparison with two other methods applied to an active suspension system, Eur. J. Control 9 (2003).
[23] K. Ito, S.S. Ravindran, A reduced basis method for control problems governed by PDEs, in: W. Desch, F. Kappel, K. Kunisch (Eds.), Control and Estimation of Distributed Parameter Systems. Proceedings of the International Conference in Vorau, 1996, 1998, pp. 153–168.
[24] T. Iwasaki, R.E. Skelton, J.C. Geromel, Linear quadratic suboptimal control with static output feedback, Syst. Control Lett. 23 (1994) 421–430.
[25] F. Jarre, A QQP-minimization method for semidefinite and smooth nonconvex programs, Tech. report, The University of Notre Dame, USA, 2000.
[26] P.P. Khargonekar, M.A. Rotea, Mixed $\mathscr{H}_2/\mathscr{H}_\infty$ control: a convex optimization approach, IEEE Trans. Automat. Control 36 (1991) 824–837.
[27] K. Kunisch, S. Volkwein, Control of Burgers' equation by a reduced order approach using proper orthogonal decomposition, J. Optim. Theory Appl. 102 (1999) 345–371.
[28] K. Kunisch, S. Volkwein, Galerkin proper orthogonal decomposition methods for parabolic problems, Numer. Math. 90 (2001) 117–148.
[29] K. Kunisch, S. Volkwein, Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics, SIAM J. Numer. Anal. 40 (2002) 492–515.
[30] S. Lall, J.E. Marsden, S. Glavaski, Empirical model reduction of controlled nonlinear systems, in: Proceedings of the IFAC Congress, vol. F, 1999, pp. 473–478.

[31] F. Leibfritz, Static Output Feedback Design Problems, Shaker Verlag, Aachen, Germany, 1998, ISBN 3-8265-4203-7.
[32] F. Leibfritz, A LMI-based algorithm for designing suboptimal static $\mathscr{H}_2/\mathscr{H}_\infty$ output feedback controllers, SIAM J. Control Optim. (2001).
[33] F. Leibfritz, *COMPl$_e$ib*: Constrained matrix-optimization Problem library—a collection of test examples for nonlinear semidefinite programs, control system design and related problems, Tech. report, Universität Trier, submitted.
[34] F. Leibfritz, E.M.E. Mostafa, An interior point constrained trust region method for a special class of nonlinear semi-definite programming problems, SIAM J. Optim. 12 (4) (2002) 1048–1074.
[35] F. Leibfritz, E.M.E. Mostafa, Trust region methods for solving the optimal output feedback design problem, Int. J. Control (2003).
[36] F. Leibfritz, E.W. Sachs, Inexact SQP interior point methods and large scale optimal control problems, SIAM J. Control Optim. 38 (1) (1999) 272–293.
[37] H.V. Ly, H.T. Tran, Modelling and control of physical processes using proper orthogonal decomposition, Math. Comput. Model. 33 (2001) 223–236.
[38] E.O. Omojokun, Trust region strategies for optimization with nonlinear equality and inequality constraints, Ph.D. thesis, University of Colorado, Boulder, 1989.
[39] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.
[40] M. Rathinam, L. Petzold, Dynamic iteration using reduced order models: a method for simulation of large scale modular systems, SIAM J. Numer. Anal. 40 (2002) 1446–1474.
[41] M. Reed, B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1980.
[42] S.Y. Shvartsman, Y. Kevrikidis, Nonlinear model reduction for control of distributed parameter systems: a computer-assisted study, AIChE J. 44 (1998) 1579–1595.
[43] L. Sirovich, Turbulence and the dynamics of coherent structures. Parts I–III, Quart. Appl. Math., XLV 22 (1987) 561–590.
[44] V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis, Static output feedback—a survey, Automatica 33 (1997) 125–137.
[45] K.Y. Tang, W.R. Graham, J. Peraire, Optimal control of vortex shedding using low-order models. I: Open loop model development. II: Model based control, Int. J. Numer. Methods Eng. 44 (1999) 945–990.
[46] L. Vandenberghe, S. Boyd, Semidefinite programming, SIAM Rev. 38 (1996) 49–95.
[47] S. Volkwein, Optimal control of a phase-field model using the proper orthogonal decomposition, Z. Angew. Math. Mech. 81 (2001) 83–97.
[48] S. Volkwein, Optimal and suboptimal control of partial differential equations: augmented Lagrange-SQP methods and reduced order modelling with proper orthogonal decomposition, Grazer Math. Berichte, Bericht Nr. 343, 2001.
[49] K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control, Prentice Hall, Upper Saddle River, NJ, 1996.
[50] K. Zhou, P.P. Khargonekar, An algebraic Riccati equation approach to $\mathscr{H}_\infty$ optimization, Syst. Control Lett. 11 (1988) 85–91.